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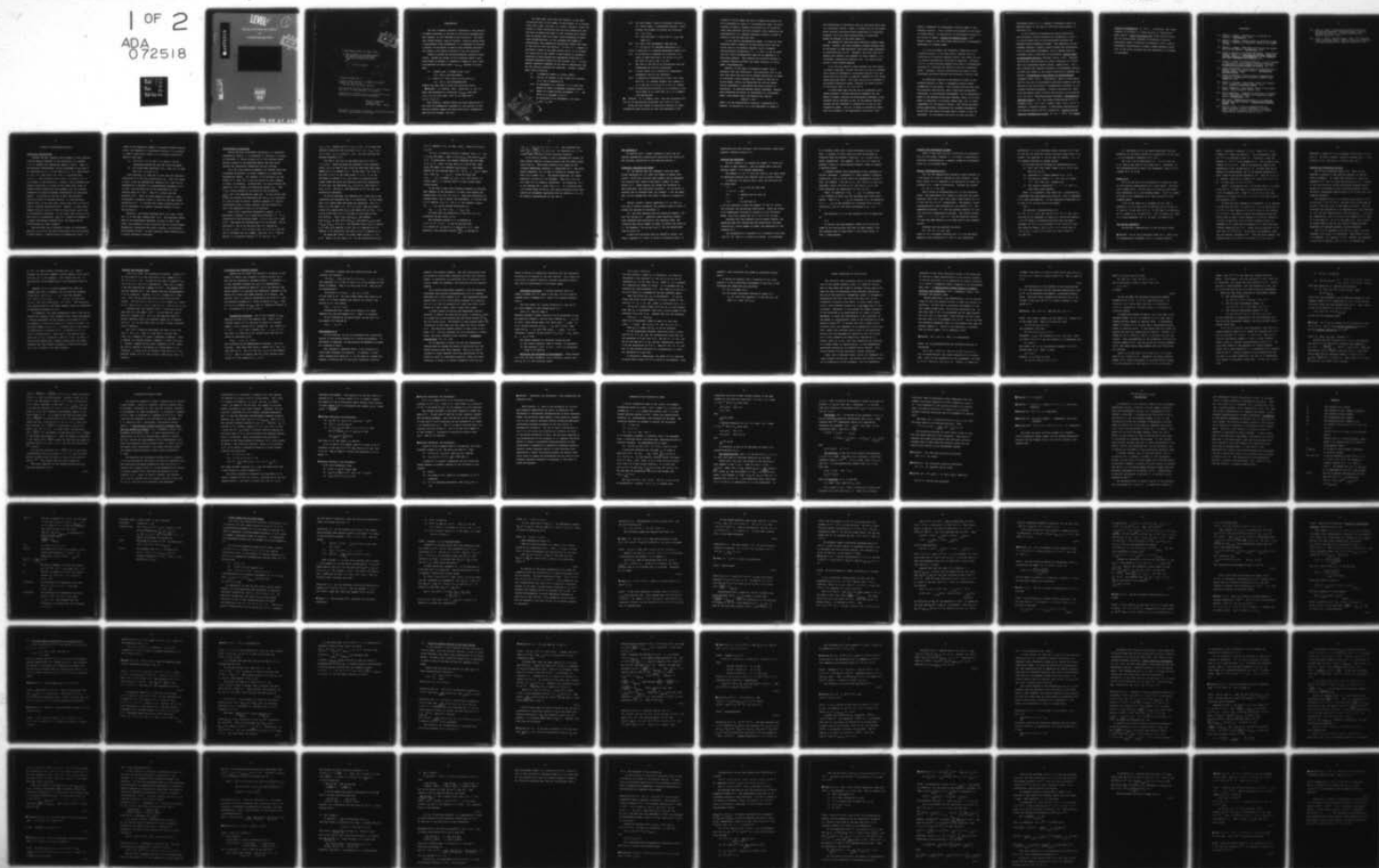
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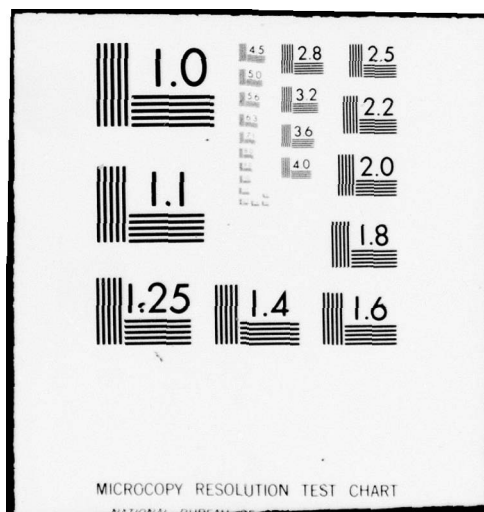
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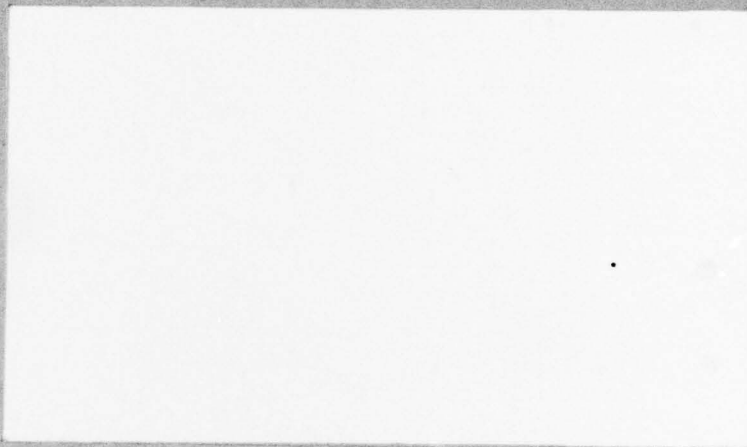
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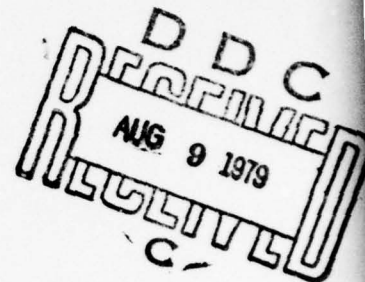


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6 A NONSTANDARD THEORY OF GAMES, PART I:

ON THE EXISTENCE OF THE QUASI-KERNEL  
AND RELATED SOLUTION CONCEPTS  
FOR \*FINITE COOPERATIVE GAMES.

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10 Alain A. Lewis\*

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## INTRODUCTION

The use of measure theoretic techniques in the analysis of games originated in the work of the Polish mathematician, Karl Urbanik, on the "Problem of Fair Division." The problem, stated briefly, runs as follows: Given an object which  $n$  participates value differently, it is possible to allocate a portion of the object to each so that everyone receives a fair share of the object according to his own valuation.

Urbanik provided the following measure theoretic formulation: Assume the object to be an abstract space  $\xi$  upon which there is defined a  $\sigma$ -algebra of subsets  $M$ , and  $n$  positive measures of finite value,  $\{u_i\}_{i \in n}$ , defined on  $M$  such that:

- (i)  $\forall_i \forall m \in M \ u_i(m) > 0 \rightarrow \exists \hat{m} \subseteq m : u_i(\hat{m}) < u_i(m)$   
i.e., the  $u_i$  are non-atomic;
- (ii)  $\exists i \exists j (i \neq j) : m \in M \ u_i(m)/u_i(\xi) \neq u_j(m)/u_j(\xi)$   
i.e., the  $u_i$  are nonproportional.

Urbanik was then able to prove the following result:

► Theorem: (K. Urbanik, 1954) Conditions (i) and (ii) imply the existence of a partition  $\{\xi_\alpha\}_{(\alpha \in n)}$  such that  $u_i(\xi_i) \geq u_i(\xi)/n$  for each  $i$  and for at least one  $j$ ,  $u_j(\xi_j) >> u_j(\xi)/n$ .

More recently, measure theory has found application in the field of mathematical economics in the analysis of the relation between competitive equilibrium and a cooperative game solution concept, the core.



For some time, since the 19th century, it had been conjectured that as the number of participants in an economy grows very large, the core, in a sense, "shrinks" closer and closer to those allocations that are just competitive ones. The work of Debreu and Scarf in 1963 verified this conjecture by showing that for large, finite economies of sufficient characteristics, as one increases the number of participants, the competitive allocations differ from those of the core by less than epsilon. In 1964, R. J. Aumann [1] published a paper on the same subject and in it he modeled the large size of the number of participants by assuming a continuum of participants and represented coalitions as Lebesgue measurable subsets of the interval  $[0,1]$ . In this measure theoretic framework he was able to demonstrate that the two solution concepts are identical. Briefly, Aumann's model runs as follows:

- (i) A commodity bundle is a point  $x \in R_+^n = \Omega$ .
- (ii) The set of traders is the closed unit interval  $[0,1]$  and is denoted by  $T$ .
- (iii) An assignment is a function  $x : T \rightarrow \Omega$  each component of which is Lebesgue integrable over  $T$ .
- (iv) There is a fixed initial assignment  $i : T \rightarrow \Omega$  for which it is assumed  $\int_T i dt > 0$ .
- (v) An allocation is an assignment  $x$  for which  $\int_T x dt = \int_T i dt$ .

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- (vi) For each trader  $t$  there is defined a relation  $\succsim_t$  on  $\Omega$  which agent  $t$ 's preference function. Preferences are assumed to satisfy the following:
  - (a)  $x \succeq y \rightarrow x \succsim_t y$
  - (b)  $\forall y \in \Omega$  the sets  $\{x : x \succsim_t y\}$  and  $\{x : y \succsim_t x\}$  are open in  $\Omega$ .
- (vii) If  $x$  and  $y$  are assignments, then the set  $\{t : x(t) \succsim_t y(t)\}$  is Lebesgue measurable in  $T$ .
- (viii) A coalition is a Lebesgue measurable subset of  $T$ .
- (ix) An allocation  $y$  dominates an allocation  $x$  if there is some coalition  $S$  for which  $y(t) \succsim_t x(t)$  for each  $t \in S$  and  $\int_S y dt = \int_S idt$ .
- (x) The core is the set of all allocations that are undominated by non-null coalitions.
- (xi) A price vector  $p$  is a vector of  $n$ -components, nonnegative and all not vanishing.
- (xii) A competitive equilibrium is a pair  $(p, x)$  such that  $x(t)$  is a maximal element with respect to  $\succsim_t$  in the set  $\{x : p \cdot x \leq p \cdot i(t)\}$  for a.e. trader.
- (xiii) An equilibrium allocation is an allocation  $x$  for which there is a  $p$  such that  $(p, x)$  is a competitive equilibrium.

► Theorem: (R. J. Aumann, 1964) The core coincides with the set of equilibrium allocations (cf. also [6,7,10]).

Since the core stands in particular relation to other cooperative game solutions as they are expressed in the

context of finite games and since a competitive market can be re-expressed in terms of a noncooperative game, the above theorem of Aumann's suggests the possibility of extending other game theoretic solution concepts, both cooperative and noncooperative, to a measure theoretic context in order to observe if and how they may vary.

In the case of noncooperative games, it turns out that extending Nash's concept of an equilibrium point from the finite case to the measure theoretic case is somewhat straightforward conceptually [9]. In fact, it can be shown that any finite noncooperative game can be embedded in a non-atomic measure. This enables one to derive results in a measure theoretic context that apply directly to finite games (noncooperative).

However, in the case of cooperative games, such a procedure runs into difficulty. The difficulty lies in the fact that while the core can be defined exclusively in terms of coalitions, other concepts that relate to the core in finite games, such as the Bargaining Set and the Kernel, require reference to individuals and their relation to coalitions. In standard measure theory (Lebesgue), however, sets consisting of points, or individuals in a non-atomic model, have measure zero; and because this implies

$$v(S \cup \{x\}) = v(S)$$

where  $v$  is the characteristic function, represented by a measure, of coalition  $S$ , it is not meaningful to speak of

the contribution or detracting that an individual might make to a coalition's worth. Hence, a concept such as the Kernel which involves inter-participant comparisons of bargaining strength, vis à vis their contributions to coalitions, cannot be defined in the usual sense [2].

Two methods of resolving this difficulty suggest themselves. Firstly, one could attempt to define concepts such as the Kernel in some equivalent form which makes reference to coalitions only [8,11]. Secondly, one could look for a measure that behaves like Lebesgue in other respects, i.e., non-atomic, translation invariant, etc., but which allows points to have nonzero measure.

It is with the second of these methods that this series is chiefly concerned. We show that with the use of a somewhat novel branch of mathematics, nonstandard analysis [4,5] measures exist that provide a conceptual framework for the second method mentioned above. This task, along with a treatment of nonstandard cooperative games, comprises the subject matter of the series, Parts I, II, III, and IV.

It would seem then that the use of nonstandard techniques provides a unified analytical framework for the Theory of Games, allowing the simultaneous treatment of the standard finite contexts as well as the measure theoretic context and the treatment of cooperative as well as non-cooperative games. One can, by means of externally imbedding the integers in the appropriate \*Finite set, even



obtain a framework for denumerably infinite games in non-standard analysis. This procedure was employed in the paper of Eugene Wesley, "An Application of Nonstandard Analysis to the Theory of Games," Journal of Symbolic Logic, 1971.

Wesley's paper is the first known application of nonstandard techniques to  $n$ -person games.

It is the extremely rich expressive capability of non-standard analysis, in the above sense, together with its conservation of existing standard results, that makes its use as a general mathematical technique so appealing. Elsewhere [12,13], we have provided examples from topics in Mathematical Economics to illustrate the method of proof by nonstandard means. Typically, \*Finite constructions are used to treat preference structures defined over discretely infinite time.

An additional feature of the method of proof by non-standard analysis is that it frequently entails weaker assumptions of set theory. This follows from the fact that the principal construction used in employing nonstandard techniques, an  $N_1$ -saturated enlargement, can be generated by means of the Boolean Prime Ideal Theorem, which, as is well known to logicians, is strictly weaker than, and, in fact, independent of, the Axiom of Choice. The latter principle having undesirable consequences for standard mathematics, proofs that can obtain the same result without it are not unwelcome. To illustrate this point, we have provided a



nonstandard proof of L. S. Shapley's fundamental result on Balanced Games, in the case of infinitely many players, in Reference [14].

In the interest of rendering the series effectively self-contained, we have extended the Introduction of Part I to include expositions of both Nonstandard Analysis and Standard Finite Cooperative Games. The exposition of nonstandard analysis "lightly" walks through the construction of an  $N_1$ -saturated enlargement, and is adapted from the treatment found in Machover and Hirschfeld, Lecture Notes on Nonstandard Analysis (Springer Verlag, 1969). Separate sections are then provided on formal properties of \*Finite sets and nonstandard measure theory, respectively. For a more rigorous and complete treatment of nonstandard analysis we refer the reader to the recent treatise by Luxemburg and Stroyan, Introduction to the Theory of Infinitessimals (Academic Press, 1976). Additionally, there is much precedent for the use of nonstandard techniques stemming from the results of the Yale School of Mathematical Economics on nonstandard exchange economies. Notable contributions, apart from the seminal works of Brown and Robinson, are contained in M. Ali Khan, "Some Equivalence Theorems," The Review of Economic Studies, 1974; the recent dissertations of R. M. Anderson (Yale, 1977) and S. Rashid (Yale, 1976); and R. M. Anderson's innovative paper, "A Nonstandard Representation for Brownian Motion and Itô Integration," Bulletin of the American Mathematical Society, 82, No. 1 (1976), and Israel

Journal of Mathematics, 25 (1976). In addition, the recent research of Professor H. Jerome Keisler in "Hyperfinite Model Theory," Logic Colloquium 1978 (North Holland Press), and "An Infinitesimal Approach to Stochastic Analysis," Preliminary Paper, University of Wisconsin (1978), employs Nonstandard Constructions to model random phenomena occurring in the social sciences and the mathematics of stochastic processes.

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## A Model of Nonstandard Analysis

### Conceptual Preliminaries

Perhaps the most commonly used argument in Real Analysis involves making reference to the continuity of a mapping,  $f: X \rightarrow Y$ , between two topological spaces,  $X$  and  $Y$ . Then, in the notation of Weierstrass, we are accustomed to say that  $f$  is continuous at  $p \in X$ , if for every  $\epsilon \in \mathbb{R}_+ - \{0\}$ , there exists a  $\delta \in \mathbb{R}_+ - \{0\}$ , such that  $|f(x) - f(p)| < \epsilon$  provided that  $|x - p| < \delta$ . The intuitive notion that " $x$  is close to  $p$ " is not present in the definition. However, every mathematician reasons intuitively in terms of one object being close to another before translating the argument into any rigorous framework of relations between neighborhoods in topological spaces. If one were to ask how to make the basically intuitive notion of nearness precise, it would minimally require that, on the real axis, points very close to zero be given a rigorous characterization. However, to treat small numbers close to zero as a form of infinitesimal would lead to contradictory reasoning. On the other hand, owing to the fact that  $\mathbb{R}$  is categorical, that is,  $\mathbb{R}$  cannot be imbedded in a larger continuously ordered field, one cannot add on infinitesimals as ideal elements.

The principal aim of Robinson's Theory of Nonstandard Analysis is to solve the above predicament using the methods of Model Theory, a subdiscipline of Mathematical Logic. In

terms of the elementary example of topology treated earlier, within the framework of Nonstandard Analysis it is possible to imbed a topological space  $X$  in an enlarged topological space  $X^*$  such that:

- (1)  $(\forall p \in X) \{x^* : x^* \in X^* \text{ and "x is near p"}\}$  can be rigorously defined and yet has intuitive appeal.
- (2) Any first-order predicate,  $F(x)$ , true in  $X$  is such that  $F(x^*)$  is true in  $X^*$ .

More precisely,  $X^*$  does not in fact have the identical properties of  $X$ , but only syntactically so. What this means is that, given any mathematical property of  $X$ , that can be expressed as a sentence in a predetermined language as  $F(X)$  is true, this sentence can be re-interpreted as a new sentence about  $X^*$ ,  $G(X^*)$ . Then  $X^*$ , in fact, has the property  $G(X^*)$ . Thus, to every property  $F$  about  $X$ , there corresponds a property  $G$  about  $X^*$  that has the same formal properties as  $F$ . It follows that formal reasoning about  $X^*$  can be performed in an exactly analogous fashion as reasoning about  $X$ .

Typically, one proves theorems about  $X$  by first "going out" to  $X^*$  and then "coming back" to  $X$ . The principal advantage of the nonstandard technique of proof is that, very often, the reasoning is more intuitive and can involve weaker assumptions, concerning the Axiom of Choice, in particular, than standard proofs. In many instances these features yield a certain expressive advantage.

### The Universe of Discourse

Before applying nonstandard techniques to a specified mathematical domain, it is necessary to construct a universe of discourse,  $U$ , which contains all of the relevant mathematical objects of the specified domain and upon which certain set theoretical operations are well defined.

Let  $V$  be a set having as members, all objects which are considered as objects, or atomic elements in a specified mathematical domain. For example, if we consider  $R$  as the domain of  $V$ , then  $V(R)$  is the set having all real numbers, at least, as elements. Alternatively, we could consider the domain of  $V$  as a family of topological spaces  $\{X_\alpha\}$ . Then  $V(\{X_\alpha\})$  would consist of all points in the collection

$\bigcup_{\alpha \in I} X_\alpha$ , for  $I$  the appropriate index set. In this latter case  $V$  need not contain the topologies for the  $X_\alpha$ , as a topology is a set of sets of points. However, members of  $V$  may be considered as sets in some contexts, and the possibility that  $V$  be comprised of sets is not excluded.

Given  $V$  for a specified mathematical domain, it is desirable to extend  $V$  to a transitive structure,  $U^0$ .  $U^0$  is transitive in the following sense: If  $x \in y$  and  $y \in U^0$ , then  $x \in U^0$ . The means for constructing  $U^0$  is as follows. For an arbitrary  $X$ , let  $UX$  be the union set of  $X$  defined as  $\{x : x \in y \dots y \in X\}$ . If  $X$  is not a set, or if it is a set but has no non-empty set as a member, then set  $UX = \emptyset$ . Then we define, in a recursive fashion,  $V^0 = V$  and for  $i \in N$ ,

$V_{i+1} = \cup V_i$ . Define next  $U^0 = \cup \{V_i : i \in \mathbb{N}\}$ .  $U^0$  is then the smallest transitive set such that  $V \subseteq U$ , i.e.,  $V_{0+1} = \cup V_0$ ,  $V_2 = \cup(\cup V_0)$ ,  $V_3 = \cup(\{U(\cup V_0)\})$ , etc., and will provide the desired extension of  $V$ .

For any  $X$ , let  $P(X)$  be the power set of  $X$ ,  $P(X) = \{x : x \subseteq X\}$ . Since non-sets are excluded in our constructions,  $x \subseteq y$  is read "x and y are two sets such that every member of x is a member of y." We say that x is a set if and only if  $x = \phi$  or for some object  $z$ ,  $z \in x$ . It is now desirable to extend  $U^0$  to a set which is transitive and closed under the power set operation and other operations. To this end, one specifies  $U_{i+1} = U_i \cup P(U_i)$  and allow  $U = \cup \{U_i : i \in \mathbb{N}\}$ . The set  $U$ , the extension of  $U^0$ , we term the universe of discourse.

It can be verified, by induction on  $i$ , that the  $U_i$  are transitive and therefore that  $U$  is transitive. We now argue that  $U$  is closed under the power set operation. Let  $x \in U$ , then  $x \in U_i$  for some  $i \in \mathbb{N}$ . If  $x$  is a set, then the transitivity of  $U_i$  means that  $x \subseteq U_i$  and hence  $P(x) \subseteq P(U_i)$ . If  $x$  is not a set then  $P(x) = \phi$ , so that in this case as well,  $P(x) \subseteq P(U_i)$ . Since  $P(U_i) \subseteq P(U_{i+1})$ , one obtains  $P(x) \subseteq U_{i+1}$ , so that  $P(x) \in P(U_{i+1})$ , and since  $U_{i+2} \subseteq U$  and  $P(U_{i+2}) \subseteq U_{i+1}$ ,  $P(x) \subseteq U$ . One observes that  $y \subseteq x \in U$  implies  $y \in U$ , and  $y \subseteq x$  implies  $y \in P(x)$  and  $x \in U$  implies  $P(x) \in U$ . However,  $U$  is transitive, and thus  $y \in P(x) \in U$  implies  $y \in U$ .

We argue next that  $U$  is closed under unions,  $\cup$ . Let  $x \in U$ . Then  $x \in U_i$  for some  $i \in \mathbb{N}$ . By the transitivity of  $U_i$ ,



$z \in y \in x$  implies  $z \in U_i$ , so that  $x \subseteq U_i$ . Hence  $\cup x \in P(U_i) \subseteq U_{i+1} \subseteq U$ .

If  $x \subseteq U$ , it does not follow in general that  $x \in U$ . But if  $x \subseteq U_i$  for some  $i$ , then  $x \in P(U_i) \subseteq U_{i+1}$  and since  $U_{i+1} \subseteq U$ ,  $x \in U$ . In particular, one readily observes that for each  $i \in \mathbb{N}$ ,  $U_i \in U$ . Also, if  $x \subseteq U$  and  $x$  is a finite set, then  $x \subseteq U_i$  for some  $i \in \mathbb{N}$ , because  $U_0 \subseteq U_1 \subseteq U_2 \subseteq \dots$ , and consequently we can conclude that  $x \in U$ . If  $x_1, \dots, x_n \in U$ , where  $n \geq 0$ , then  $\{x_1, \dots, x_n\} \in U$ . Since the union set  $\cup\{x_1, \dots, x_n\}$  is simply  $x_1 \cup x_2 \cup \dots \cup x_n$ , and since  $U$  is closed under  $\cup$ , it follows that  $U$  is also closed under finite unions.

If we wish to deal with Cartesian products of finitely many members of the universe  $U$ , we must first define the ordered pair  $(x, y)$  as  $\{\{x\}, \{x, y\}\}$ . From this definition of ordered pairs, due to Weiner and Kuratowski, it follows that  $x, y \in U$  implies  $(x, y) \in U$ . For  $n > 2$ , the ordered  $n$ -tuple,  $(x_1, \dots, x_n)$  can be defined recursively as

$$(x_1, \dots, x_n) \text{ df } = ((x_1, \dots, x_{n-1}), x_n)$$

To allow the last definition to hold for  $n = 2$ , by convention we may allow  $(x) = x$ .

The Cartesian product  $X \times Y$  is defined as  $\{(x, y) : x \in X \dots y \in Y\}$ . It can be verified that  $X \times Y \in \text{PPP}(X \cup Y)$  so that  $X, Y \in U$  implies  $X \times Y \in U$ . More generally, the Cartesian product  $\prod_{i=1}^n Y_i$  is defined as

$\{(y_1, \dots, y_n) : y_i \in Y_i \text{ } i=1, \dots, n\}$ . One therefore has that  $\prod_{i=1}^n Y_i = \left( \prod_{i=1}^{n-1} Y_i \right) \times Y_n$  for  $n \geq 2$ . By induction on  $n$ ,  $U$  is closed under arbitrary finite Cartesian products.

If we wish to convert  $U$  into a mathematical system, we must further specify a binary relation and two binary operations over it. The binary relation will simply be the ordinary membership relation,  $\epsilon$ , restricted to  $U$ . The first binary operation,  $pr$ , is that of forming an ordered pair. Then  $x \text{ } pr \text{ } y$  means  $(x, y)$ . The second operation,  $ap$ , means applying a function to its argument. Then for  $f \text{ } ap \text{ } x$  we will mean  $f(x)$  as commonly understood. By convention, if  $f$  is not defined for  $x$ , then  $f \text{ } ap \text{ } x = \phi$ . It is obvious that  $U$  is closed under  $ap$  and  $pr$ , and their inclusion allows such notions as arithmetic operations such as addition,  $x+y$ , to be formally characterized as  $+ap \{xpr y\}$ .

### The Language, L

We specify next a formal language in which one can express mathematical propositions concerning the objects of the universe constructed in the previous section, U.

### Constants, variables and terms

(a) We require that the language L have for each object belonging to U at least one symbol to denote that object. While only finitely many symbols can be expressed, one requires that there be available a symbol for each object in U. These symbols are termed the constants or, more precisely, the individual constants. If the object in question has an accepted name, for example,  $\emptyset$  for the empty set, we will assume that this name is used as a constant in L.

Certain letters, usually uppercase x's, y's and z's, are used to denote variables, the intended range of which is always the universe of discourse, U.

(b) One also requires that the operation symbols, "pr" and "ap" belong to L. Combining these operation symbols with constants and variables in the ordinary sense, and iterating any finite number of times, we obtain the terms of L. For example, "cos ap{+ap{ $\pi$  pr x}}" has the abbreviated form of  $\cos(\pi + x)$ .

A term with variables does not denote an object, but when a constant or a term, in which no variables occur, is

substituted for each variable, then one obtains a term which denotes a definite object in  $U$ .

### Formulae and sentences

We will require  $L$  to possess the symbol " $=$ " which will be taken to mean identity. Also we assume that  $L$  has the relation symbol " $\epsilon$ " to denote membership.

The symbols " $=$ " or " $\epsilon$ " with two terms in the usual sense of appearance generate the simplest form of atomic formula.

In addition, we require that  $L$  have the following set of connectives:

" $\neg$ "	- it is not the case that
" $\wedge$ " or " $\&$ "	- and
" $\Rightarrow$ "	- implies that or only if
" $\vee$ "	- or
" $\Leftrightarrow$ "	- if and only if

$L$  is also required to have the symbols " $\forall$ " and " $\exists$ " called the universal and existential quantifiers. These are always to be immediately followed by variables in the following sense,  $(\forall x)(F \text{ ap } x)$  means for all  $x$  (in  $U$ )  $F(x)$  is true.

Starting with the simplest form of atomic formula, and combining them with connectives and quantifiers, and then iterating any finite number of times, one generates all the formulas of  $L$ .

Any occurrence of a variable  $x$  in a context of the form " $(\forall x)(\phi)$ " or " $(\exists x)(\phi)$ " is said to be bound. An occurrence



of a variable other than a bound occurrence is said to be free. A formula in which there are free occurrences of variables does not express a definite, i.e., either true or false, proposition. For example, " $(\forall y) (\neg(y \in x))$ " does not signify a true or a false meaning because  $x$  does not denote a definite object.

A formula without free occurrences of any variables is called a sentence. A sentence of  $L$  does express a definite proposition, true or false, about the mathematical system  $U$ . Then, for example, " $(\exists x) (\forall y) (y \in x)$ " and " $(\forall y) (y \in \phi)$ " are true sentences, while " $(\forall x) (\forall y) (y \in x)$ " and " $(\forall y) (y \in (\phi, \phi))$ " are false sentences (we substitute  $y \notin x$  for  $\neg(y \in x)$ ).

Often, we symbolize " $\phi(x_1, \dots, x_n)$ " to indicate that  $\phi$  is a formula in which only the variables have free occurrences. Then if  $c_1, \dots, c_n$  are constants in  $L$ , we symbolize " $\phi(c_1, \dots, c_n)$ " as the sentence of  $L$  obtained from  $\phi$  by substituting  $c_i$  for all the free occurrences of  $x_i$ ,  $i=1, \dots, n$ .

The assertion " $\phi$  is a true sentence" will be symbolized as

$$\models \phi$$

Of course, the knowledgeable reader will not be confused by the distinctions that must be made between  $L$  and the language used to speak about  $L$ , the latter being, in fact, a metalanguage.

### Standard and Nonstandard Systems

There is more than one way of providing an interpretation for the formal language,  $L$ . In order to characterize arbitrary interpretations, a general scheme for interpretations of  $L$  is provided below.

### General interpretation of $L$

Let  $I$  be the mapping which assigns to each constant in  $L$  the object in  $U$  that is denoted by that constant. Since every object in  $U$  is supposed to be denoted by at least one constant of  $L$ ,  $I$  must be surjective. Consider the system

$$M = (U, I, \epsilon, \text{pr}, \text{ap})$$

Then  $M$  gives an interpretation of  $L$  in the following sense. The quantifiers of  $L$  refer to  $U$  and have the interpretation that " $(\exists x)$ " and " $(\forall x)$ " mean "there exists an  $x$  in  $U$ " and "for all  $x$  in  $U$ ," respectively. The mapping  $I$  gives interpretation to the constants of  $L$  as objects in  $U$ . The membership relation, " $\epsilon$ ", and the operator symbols " $\text{pr}$ " and " $\text{ap}$ " are to be interpreted as defined. The logical connectives, " $\wedge$ ", " $\vee$ ", " $\Rightarrow$ ", and " $\rightarrow$ ", and the equality relation " $=$ " have the same meaning in  $M$  as they have in any interpretation.

Consider next the abstract structure,

$$M^* = (U^*, I^*, \epsilon^*, \text{pr}^*, \text{ap}^*)$$

where  $U^*$  is a non-empty arbitrary set;  $I^*$  is an arbitrary mapping of the constants of  $L$  into  $U^*$  (not necessarily

surjective);  $\epsilon^*$  is an arbitrary binary relation on  $U^*$ ; and  $pr^*$  and  $ap^*$  are arbitrary binary relations under which  $U^*$  is closed. The asterisk "\*" can be read as "pseudo," i.e.,  $M^*$  is a pseudo structure or pseudo system.

It is clear that  $M^*$  provides an interpretation of  $L$ :

- (1) " $(\exists x)$ " and " $(\forall x)$ " mean "there is an  $x$  in  $U^*$ " and "for all  $x$  in  $U^*$ ."
- (2) Constants of  $L$  denote members of  $U^*$ ,  $I^*(c)$ .
- (3) The relation symbols " $\epsilon$ ", " $pr$ " and " $ap$ " denote " $\epsilon^*$ ", " $pr^*$ " and " $ap$ ", respectively.
- (4) The logical connectives " $\wedge$ ", " $\vee$ ", " $\Rightarrow$ " and " $\rightarrow$ " have their usual meaning.

When  $\phi$  is a sentence, then under the  $*$  interpretation,  $\phi$  expresses the same proposition about  $M^*$ , that it expresses in  $M$ , when reinterpreted. If the proposition expressed by  $\phi$  is true in  $M^*$ , we say this by the symbol

$$* \models \phi$$

Two distinct interpretations of  $L$ ,  $M^* = (U^*, I^*, \epsilon^*, pr^*, ap^*)$  and  $\bar{M}^* = (\bar{U}^*, \bar{I}^*, \bar{\epsilon}^*, \bar{pr}^*, \bar{ap}^*)$ , are said to be isomorphic if there is a one-to-one mapping  $\psi: U^* \rightarrow \bar{U}^*$  which is surjective and such that for every constant " $c$ " of  $L$  it is true that  $\bar{I}^*(\bar{c}) = \psi(I^*(c))$  and such that for every  $\alpha$  and  $\beta$  in  $U^*$ ,  $\alpha \epsilon^* \beta$  if and only if  $\psi(\alpha) \bar{\epsilon}^* \psi(\beta)$ ,  $\psi(\alpha pr^* \beta) = \psi(\alpha) \bar{pr}^* \psi(\beta)$ , and  $\psi(\alpha ap^* \beta) = \psi(\alpha) \bar{ap}^* \psi(\beta)$ .

If  $\psi$  satisfies all of the above conditions with the exception of its being surjective a possibility, we say that  $\psi$  is an isomorphic imbedding of  $M^*$  in  $\tilde{M}^*$ .

Let  $S$  be a set of sentences of  $L$ . If  $* \models \Sigma$  for all  $\Sigma \in S$ , then we say that  $M^*$  is a model for  $S$ . Obviously, if two interpretations  $M^*$  and  $\tilde{M}^*$  are isomorphic, then if  $M^*$  is a model of  $S$ , so is  $\tilde{M}^*$ .

### Models of $K$

Let  $K$  be the set of all sentences of  $L$  which are true in the original interpretation of  $L$  given above. Trivially,  $M$  is a model for  $K$ . We will call  $M$ , as well as any structure isomorphic to it, the standard model. If  $M^*$  is a model of  $K$  but is not isomorphic to  $M$ , then we call  $M^*$  a non-standard model.

If  $M^*$  is any model of  $K$ , standard or not, then a sentence  $\Sigma$  of  $L$  is true in  $M^*$  if and only if it is true in  $M$ , that is, if and only if  $\Sigma \in K$ . If  $* \models \Sigma$  then  $\Sigma \in K$  and since  $M^*$  is a model of  $K$ , we have  $* \models \Sigma$ . If  $\Sigma$  is false in  $M$ , then  $\models \rightarrow \Sigma$  and therefore  $* \models \rightarrow \Sigma$ , so that  $\Sigma$  is false in  $M^*$ .

### The natural imbedding

We provide a demonstration of the following result.

► **Theorem:** Let  $M^*$  be an arbitrary model of  $K$ . Then  $M$  can be isomorphically imbedded in  $M^*$  in a unique fashion.



Proof: Consider a constant "c" of L. Then  $I("c")$  is an element of U under the mapping I. For convenience's sake, let  $I("c")$  be denoted as just "c". Similarly, under the mapping  $I^*$ ,  $I^*("c")$  is an element of  $U^*$ . Let us denote this element of  $U^*$  as "c\*". Then if we desire to imbed M isomorphically in  $M^*$ , we must put  $c \mapsto c^*$ . To show that such a mapping is single-valued, let "d" be another constant of L that I maps on the object denoted by "c". Then  $\models (c = d)$  in M. However, since  $M^*$  is a model of K, then  $\models (c^* = d^*)$  in  $M^*$ . In that case,  $I^*("c") = I^*("d")$ .

Since I is surjective on U, it follows that  $I^*$  is injective into  $U^*$ . To show that  $I^*$  is one-to-one, let c, d,  $(c \neq d)$  be members of U. Then  $\models \rightarrow (c = d)$  in M, but then since  $M^*$  is a model of K,  $\models \rightarrow (c^* = d^*)$  in  $M^*$  and  $c^*$  and  $d^*$  are distinct in  $U^*$ .

To show that the imbedding is isomorphic, it is required to show that for any c, d, and e in U,  $c \in d$  is equivalent to  $c^* \in^* d^*$ ,  $e = c \text{ prd}$  implies  $e^* = c^* \text{ pr}^* d^*$  and  $e = c \text{ apd}$  implies  $e^* = c^* \text{ ap}^* d^*$ , the proofs of which are elementary exercises. Notice that we use  $e = c \text{ prd}$  implies  $e^* = c^* \text{ pr}^* d^*$  instead of  $e = c \text{ prd}$  if and only if  $e^* = c^* \text{ pr}^* d^*$  because since M is imbedded in  $M^*$ , there may be elements in  $U^*$  such that their inverse images are not in U. There will be constants  $t^* \in U^*$  such that  $(I^{*-1}("t^*")) \notin U$ . As we shall see presently these are the external elements of  $M^*$ . From the above remarks, one observes that M is onto  $M^*$  if and only if  $I^*$  is onto  $U^*$ .

Therefore, a model  $M^*$  of  $K$  is standard if and only if  $I^*$  is onto  $U^*$ . We shall, however, implicitly assume that  $M$  is an extension of  $M$ , from which it follows that  $U \subseteq U^*$ .

### Standard and nonstandard objects

Let  $M^*$  be an arbitrary model of  $K$ . Then, as was remarked in passing in the previous section,  $U^*$  has two kinds of members: (a) those that belong to  $U$ , since  $M$  is imbedded in  $M^*$ , and (b) those that belong to  $U^* - U$ . Objects of the first kind are called by Robinson Standard objects objects of the second kind are called Nonstandard objects. Obviously, nonstandard objects exist in  $U^*$  if and only if  $M^*$  properly extends  $M$ .

For standard  $a$  and  $b$ , it is safe to say that  $\models (a \in b)$  if and only if  $\models^* (a^* \in^* b^*)$ , and that  $a \text{ pr } b$  is the same as  $a^* \text{ pr }^* b^*$ , and that  $a \text{ ap } b$  is the same as  $a^* \text{ ap }^* b^*$ . This is obviously true because  $M^*$  is an extension of  $M$ , and therefore  $\in^*$ ,  $\text{pr}^*$ , and  $\text{ap}^*$  restricted to  $U$  are the same as  $\in$ ,  $\text{pr}$ , and  $\text{ap}$ , respectively, and if  $a^*$  and  $b^*$  are standard in  $U^*$ , then  $a^*$  is the same as  $a$ , as is  $b^*$  the same as  $b$ . Also, since  $U$  is closed under  $\text{pr}$  and  $\text{ap}$ , the result of applying these operations to standard objects is always standard.

Now, if  $S$  is standard in  $M^*$  and  $\alpha \in S$ , then  $\alpha$  is also standard in  $M^*$  since  $U$  is transitive. Then since  $S^* = S$  if  $S$  is standard,  $\alpha \in S$  implies  $\alpha^* \in^* S^*$ . However, the converse is not true in general, that is, if  $S$  is standard and

$a^* \in S^*$ , we cannot safely conclude that  $a \in S$ . Then a standard set may have \*members (pseudo members) which therefore are not in fact members. This simple fact can, in applications, give augmentation to the expressive power of  $M^*$ .

However, if  $S$  is a finite standard set, then its \*members are just its members, all of which are standard. To see this, let  $S = \{a_1, \dots, a_n\}$ , then the sentence  $\phi(x) \equiv (\forall x) x \in S \Leftrightarrow (x = a_1 \dots \dots x = a_n)$  is such that  $\models \phi(x)$ , then also since  $M \subset M^*$ ,  $\models \phi^*(x^*)$ . The difficulty with infinite sets is that their members cannot be enumerated by sentences analogous to  $\phi(x)$ .

To summarize, for every mathematical notion that can be defined for objects of the standard world, there corresponds a \*notion meaningful for the objects in any model  $M^*$  of  $K$ . For example, a \*set is an object in  $U^*$  which is either  $\phi$  or has some \*member ( $I^*(\phi) = \phi$ ). A \*ordered pair is the result of applying  $pr^*$  to objects of  $U^*$ . A relation in  $U^*$  is a \*set of \*ordered pairs. Since  $M \subset M^*$ , one observes that every set in  $U$  is a fortiori a \*set in  $U^*$  etc. We shall in what follows employ the \* notation freely with the understanding that the context of meaning is clear and we shall use  $\in$  and  $=$  without \* for both contexts.

### Internal and external sets

Let  $S$  be a \*set, not necessarily standard. Define  $\hat{S}$  to be the scope of  $S$  as the collection of all \*members of  $S$ . So,  $\hat{S} = \{\alpha \in U^* : \alpha \in S\}$ . It must be said that in general  $\hat{S}$  is not a set in  $U$ , nor is it necessarily a \*set, but is simply a set well defined and a member of  $P(U^*)$ . The members of  $\hat{S}$  are the \*members of  $S$ . One remarks that the correspondence  $S \rightarrow \hat{S}$  is injective. Further, if  $t$  is a \*subset of  $S$ , then  $\hat{t} \subseteq \hat{S}$ . This follows from the truth of the following sentences:  $(\forall x)(\forall y)(x \subseteq y \Rightarrow (\forall z)(z \in x \Rightarrow z \in y))$  where  $x \subseteq y$  means  $\{(\phi = x \dots (\exists z)(z \in x)) \dots (\phi = y \dots (\exists z)(z \in y)) \dots (\forall z)(z \in x \Rightarrow z \in y)\}$ .

Consider next, any subset  $T$  of  $\hat{S}$ . It may well be the case that  $T = \hat{t}$  for some  $t$  in  $U^*$ . If this were so, then  $t$  must be a \*subset of  $S$ . If it is the case that such a  $t$  exists, Robinson calls  $T$  an internal subset or an internal set of  $S$ . If, on the other hand, no such  $t$  exists, Robinson calls  $T$  external.

Let  $f$  be a \*function which \*maps the \*set  $A$  \*into the \*set  $B$ . Let  $\alpha \in \hat{A}$ . Then  $\alpha \in A$  and then it follows that  $f \text{ ap }^* \alpha$  is a \*member of  $B$  since  $f$  performs a \*mapping. Thus,  $f$  induces in a natural manner a mapping  $\alpha \mapsto f \text{ ap }^* \alpha$  of  $\hat{A}$  into  $\hat{B}$ . However, suppose we are given a mapping  $\psi : \hat{A} \rightarrow \hat{B}$ . Then  $\psi$  may not be induced in the manner of  $f$  above where  $f$  is a \*function.  $\psi$  fails to be so induced if and only if  $\psi$  is an external subset of  $\hat{A} \times \hat{B}$ , and in such a case we say that  $\psi$  is external.



### A criterion for internal subsets

Suppose now we extend the language  $L$  by adding to the supply of symbols new constants to denote objects in  $U^*$  that are not necessarily standard objects. A sentence,  $\psi$ , in this extended language can still be interpreted as stating some proposition about  $M^*$ , but to be precise, one should extend  $I^*$  to map each new additional constant on the object of  $U^*$  which the added constant is to denote. The notation " $* \models \psi$ " is used to characterize the truth of  $\psi$  over the extended language of  $L$ , which we denote as  $L_{\text{ext}}$ . Then if  $\phi(x)$  is a formula of  $L_{\text{ext}}$ , and if  $\alpha$  denotes an object in  $U^*$ ,  $\alpha$  satisfies  $\phi(x)$  in  $M^*$  if  $\models \phi(\alpha)$ .

Internality criterion: Let " $S$ " be a symbol of  $L_{\text{ext}}$  denoting as  $*\text{set}$  in  $U^*$  and suppose  $T \subseteq \hat{S}$ . Then  $T$  is an internal subset of  $\hat{S}$  if and only if there is a natural number  $n$ , and  $n$  objects of  $U^*$ , denoted by, say, symbols in  $L$ , " $\beta_1$ ", ..., " $\beta_n$ ", and a formula  $\phi(y_1, \dots, y_n, x)$  in  $L$  such that  $T$  is the set of all objects such that the following is satisfied in  $M^*$ :

$$\phi(\beta_1, \dots, \beta_n, x) \dots x \in S$$

The criterion can be demonstrated as follows. Let  $T$  be internal. Then there must exist a  $*\text{subset}$  of  $S$ , say, " $\beta_1$ ", such that  $\hat{\beta}_1 = T$ . Let  $n=1$  and let  $\phi(y_1, x)$  be the formula " $x \in y_1$ ". Then  $T$  is exactly the set of all objects satisfying in  $M^*$  the formula  $(x \in \beta_1 \dots x \in S)$ .

Conversely, suppose that the condition holds, and consider the sentence:

$$(\forall z) (\forall y_1) \dots (\forall y_n) (\exists v) (\forall x) \left[ x \in v \Leftrightarrow (\phi(y_1, \dots, y_n) \dots x \in z) \right]$$

This sentence is in  $L$  and is true in  $M$ , as an instance of the Axiom of subsets. Then it is also true in  $M^*$ . Then by way of instantiation,

$$(\exists v) (\forall x) \left[ x \in v \Leftrightarrow (\phi(\beta_1, \dots, \beta_n, x) \dots x \in S) \right]$$

is also true in  $M^*$ . But this simply means that there is an object in  $U^*$  whose \*members are exactly the objects that satisfy in  $M^*$ , the formula

$$\phi(\beta_1, \dots, \beta_n, x) \dots x \in S$$

Alternatively put, there is an object in  $M^*$  whose \*members are just the members of  $T$ . Then  $T$  is internal.

If the internality criterion holds, we say that  $T$  is the subset of  $\hat{S}$  defined by

$$\phi(\beta_1, \dots, \beta_n, x)$$

### Enlargements of $M$

In this section, we will be concerned with a particular variety of nonstandard models of  $M$ , called enlargements. In the manner of Robinson, we characterize enlargements by means of a theorem of Logic.

The language  $L$ , described above, is an instance of a first-order language with equality. In general, a first-order language with equality,  $\tilde{L}$ , is the same as  $L$  except for perhaps a differing set of individual constants, relation

symbols, and operator symbols. The only restrictions that are placed on first-order languages are that each relation symbol, or predicate, be associated with an arbitrary, but finite, number of arguments, and similarly for the operator symbols.

For a given first-order language  $\bar{L}$ , one can associate structures that provide interpretations of  $\bar{L}$  in a manner described for  $L$  with respect to  $M^*$ . The Compactness Theorem asserts that if  $\bar{L}$  is a first-order language with equality, and if  $S$  is a set of sentences of  $\bar{L}$  such that every finite subset of  $S$  has a model, then  $S$  itself has a model.

A direct method of proving the Compactness Theorem consists in taking, for each finite  $s_0 \subseteq S$ , a system  $M_{s_0}$  which is a model for  $s_0$  and constructing the direct product of  $M_{s_0}$  for all finite  $s_0 \subseteq S$ . The product is then reduced modulo the ultrafilter on the index set that names the finite subsets of  $S$ . The resulting reduced product is then shown to be a model of  $S$ . The key reference for this procedure is Frayne, Morel, and Scott, "Reduced Direct Products," Fundamenta Mathematicae, Vol. 51, 1962.

It is important to point out that the Compactness Theorem does not assume the Axiom of Choice but only the weaker Boolean Prime Ideal Theorem. This is an item of interest in later chapters involving applications of the method of proof by nonstandard analysis, where the chief technique is usually to replace arguments involving the

Axiom of Choice or Transfinite Induction with the concurrent relations to be defined in the next section. This issue will also arise in measure theoretic contexts concerning measurable sets of participants in non-atomic games.

Concurrent relations: A binary relation over  $U$  is simply a subset of  $U \times U$ , which is to say, if  $R$  is a set of ordered pairs of members of  $U$ , then  $R$  is a binary relation over  $U$ .

The left domain of a binary relation  $R$  is the set of all first members of the ordered pairs in  $R$ ,

$$\text{LD}(R) = \{a : aRb \text{ for some } b\}$$

Robinson defines a binary relation  $R$  to be concurrent if for any positive integer  $n$ , and for any  $n$  objects  $a_1, \dots, a_n$  all in  $\text{LD}(R)$ , there exists some  $b$  for which  $a_1Rb, \dots, a_nRb$ . If  $R$  is a binary relation and  $a_1, \dots, a_n$  are in  $\text{LD}(R)$ , then there are  $b_1, \dots, b_n$  such that  $a_1Rb_1, \dots, a_nRb_n$ . If  $R$  is concurrent, however, we may choose  $b = b_1 = b_2 = \dots = b_n$ , given  $a_1, \dots, a_n$  in  $\text{LD}(R)$ .

Two simple examples of concurrent relations are:

- (1) any linear ordering, weak or strong, is concurrent.
- (2) the relation  $\{(a,b) : a, b \in U \dots a \neq b\}$  is concurrent, since  $U$  is infinite.

Definition and existence of enlargements: Every formula of  $L$  with two free variables,  $\phi(x,y)$  defines a binary relation  $R_\phi$  on  $U$  in the following sense:



$$R_\phi = \{(a, b) : \models \phi(a, b)\}$$

For each formula  $\phi$  whose  $R_\phi$  is concurrent, let there be designated a new constant " $c_\phi$ " and let  $K_\phi$  be the set of sentences in  $L$  of the form  $\phi(a, c_\phi)$ , where " $a$ " is a constant of  $L$  denoting an object in  $\mathcal{D}(R_\phi)$ . It is assumed that  $K_\phi$  contains all such sentences in  $L$ . Note that  $K_\phi$  is not in  $L$  but in  $\bar{L}$  which is  $L$  extended by the new constants added.

Next let  $\bar{K} = KU\{U\{K_\phi : R_\phi \text{ is concurrent}\}\}$ . It can be shown that every finite subset  $K_0$  of  $\bar{K}$  has a model of the form  $M_0 = (U, I_0, \varepsilon, pr, ap)$ , where  $I_0$  coincides with  $I$  in  $L$ .

Let  $K_0$  be a finite subset of  $\bar{K}$  and if  $\phi$  is any formula such that  $R_\phi$  is concurrent, then only a finite number of the sentences of  $K_\phi$  are in  $K_0$ . Suppose they are the sentences:

$$\phi(a_1, c_\phi), \dots, \phi(a_n, c_\phi)$$

Since  $R_\phi$  is concurrent, there is some  $b \in U$  such that  $a_1 R_\phi b, \dots, a_n R_\phi b$ . Let  $I_0("c_\phi") = b$ , and let  $I_0 = I$  on  $L$ .

Then  $M_0$  is a model for  $K_0$ , as can be verified.

By the Compactness Theorem, there must exist a structure  $\bar{M} = (\bar{U}, \bar{I}, \bar{\varepsilon}, \bar{pr}, \bar{ap})$  for interpreting  $\bar{L}$  such that all the sentences of  $\bar{K}$  are true in  $\bar{M}$ . Now let  $U^*, \varepsilon^*, pr^*$ , and  $ap^*$  be the same as  $\bar{U}, \bar{\varepsilon}, \bar{pr}$ , and  $\bar{ap}$ , respectively, and let  $I^*$  be the restriction of  $\bar{I}$  to  $L$ . Then  $M^* = (U^*, I^*, \varepsilon^*, pr^*, ap^*)$  is the desired structure for interpreting  $L$  in which all sentences of  $K$  are true.

In Robinson's terminology, any model  $M^*$  of  $K$  obtained as above from a model  $\bar{M}$  of  $\bar{K}$  is called an enlargement. More

recently, such structures are termed  $N_1$ -saturated enlargements.

It should be apparent that a structure  $M^*$  for interpreting  $L$  is an  $N_1$ -saturated enlargement if and only if the following two conditions are satisfied:

- (i)  $M^*$  is a model of  $K$ ;
- (ii) For every concurrent relation  $R_\phi$  there is a  $c_\phi \in U^*$ , such that whenever  $a \in U$  and  $\models \phi(a, b)$  for some  $b \in U$ , then  $\models \phi(a, c_\phi)$ .

### Formal Properties of \*Finite Sets

The intuition that originally led us to the consideration of the issues treated in Part I in terms of \*Finite sets is that among the class of internal \*Finite sets in  $N^*$ , there are sets of exceedingly large standard cardinality. The use of the adjective "exceedingly large" is intended to characterize the fact that while such sets are, by the internality criterion given above, in every formal sense, finite, their standard cardinality is at least that of the continuum. Therefore, if one accepts the Aumann framework of the continuum as an approximation of a model of social phenomena, in which the relationship of the individual to the masses is insignificant, owing to the cardinality of the masses as being  $2^{N_0}$ , it follows that such a model can be obtained within the framework of an appropriate \*Finite set. Such a model thus obtained carries with it the additional feature that the mathematical operations that are carried out in that framework are essentially combinatoric in nature, which is to say, that such operations are identical in a formal sense (or syntactically), to the mathematical operations carried out on standard finite objects. This latter feature follows from their internal character.

Thus, with a combined stroke, within the framework of a large internal \*Finite set, one obtains the syntactic facility of essentially finite arguments, with the expressive

character of the large cardinality given in the continuum. We provide a formal demonstration of this crucial property of \*Finite sets which is an adaptation of results contained in E. Zakon's fundamental paper, "Remarks on the Nonstandard Real Axis," in Applications of Model Theory to Algebra, Analysis, and Probability, W.A.J. Luxemburg, editor, Holt, Rhinehart and Winston, 1969.

Before embarking on the demonstration, let us convince ourselves that the objects we wish to treat are actually in existence. We know that  $N_1$ -saturated enlargements of the real number system exist, for we have constructed one. We also know that if  $M^*$  is  $N_1$ -saturated and for some standard set in  $M$ , say,  $S$ , then  $S \subseteq S^*$ , which follows trivially from the fact that if  $S$  is standard, then  $S = S^*$ . Let us now make the innocuous assumption that in  $L$  are symbols that name the natural numbers,  $N$ . Then  $N \in U$  and therefore is standard. From the above remark, in  $M^*$  therefore,  $N \subseteq N^*$ .

►Theorem: Let  $\omega \in N^* - N$ . Then  $\omega$  is nonstandard.

Proof:  $M^*$  is  $N_1$ -saturated and the following relation is concurrent:

$R_\phi = \{(x, y) : \models \phi(x, y)\}$ , for  $x, y \in N$  and  $\phi(x, y) = x < y$  for, as noted earlier, any linear ordering in  $M$  is concurrent. Then, by the properties of the enlargement, for some  $c_\phi$  in  $M^*$ ,  $\models \phi(x, c_\phi)$  for all  $x \in N$ . Clearly,  $c_\phi$  cannot be



standard, for then  $c_\phi \in N$  and it would follow that  $\models \phi(x, c_\phi)$  for all  $x \in N$ , which is clearly false of  $M$ . Then  $c_\phi$  must be in  $N^*$ .

Q.E.D.

In particular, by the theorem, we may conclude that  $N^* - N \neq \emptyset$  and therefore provide concrete demonstration that in  $M^*$ , an  $N_1$ -saturated enlargement, nonstandard objects exist. The next result sharpens the previous one in the context of the fact that  $N^*$  is linearly ordered in like manner to  $N$ .

► Theorem: Let  $\omega \in N^* - N$ . Then for any  $n \in N$ ,  $n < \omega$ .

Proof: Per contra, assume  $\omega \leq n$  for some  $n \in N$ . Choose  $\hat{n}$  to be the least such. Then it is true of  $M$  that

$$\models (\forall n \in N) (n \leq 0 \Rightarrow n = 0)$$

and therefore in  $M^*$  that

$$*\models (\forall n^* \in N^*) (n^* \leq 0 \Rightarrow n^* = 0)$$

Therefore  $\neg(\omega \leq 0)$  and therefore  $\neg(\hat{n} = 0)$  are true. Then, for some  $m \in N$   $\hat{n} = m + 1$  and one obtains as a consequence that

$$m < \omega \leq m + 1$$

But since  $\omega \in N^* - N$ , by the previous theorem,  $\neg(\omega = m + 1)$ , because then  $\omega \in N$ . Then, in fact,

$$m < \omega < m + 1$$

However, it is true of  $M$  that

$$\models (\forall n \in N) \neg (\exists m \in N) (m < n < m + 1)$$

Then it is also true of  $M^*$  that

$$*||= (\forall n^* \in N^*) \rightarrow (\exists m^* \in N^*) (m^* < n < m^* + 1)$$

and in particular, since  $m \in N \subseteq N^*$ , and  $\omega \in N^*$

$$m < \omega < m + 1$$

is false.

Q.E.D.

We can now make the following distinction between elements of  $N^*$  in  $M^*$ . If  $n \in N^*$ , then we say that  $n$  is finite if  $n \in N$ , and we shall say that  $n$  is infinite if  $n \in N^* - N$ . Generically, we shall employ  $\omega$  to indicate an infinite integer.

To generalize matters slightly, let us say that a non-standard number,  $x^* \in R^*$ , is S-finite if it is less than a standard natural number in absolute value. A nonstandard number is \*Finite if it is less than a nonstandard natural number in absolute value. In the latter case we include the possibility that the integer may be infinite. Clearly, if a number  $x^* \in R^*$  is S-finite, then it is \*Finite, but not conversely. Then corresponding to the categories of finite and infinite members of  $N^*$ , one has finite and infinite members of  $R^*$ , the latter being those numbers that are \*Finite but not S-finite.

Since it is true that  $R$  is a field in  $M$ , it is true that  $R^*$  is a \*field in  $M^*$ . Therefore, for every infinite member of  $R^*$ , there corresponds its multiplicative inverse. It is not difficult to see that if  $x^* \in R^*$  is an infinite

number, then  $(x^*)^{-1}$  is less than any standard positive number. Such numbers, with the exclusion of zero, which has no reciprocal, are termed the infinitesimals in  $M^*$ . Clearly, the set of infinitesimals, with the exception of zero, are all in  $M^*$  and none can be in  $M$ , hence they are nonstandard.

There are then three varieties of nonstandard numbers:

- (1) The infinitesimals,  $M_1$

$$M_1 = \{x^* \in R^* : |x^*| < r \text{ for any } r \in R_+ - \{0\}\}$$

- (2) The S-finite nonstandard numbers,  $M_0$

$$M_0 = \{x^* \in R^* : |x^*| < r \text{ for some } r \in R_+ - \{0\}\}$$

- (3) The infinite nonstandard numbers,  $R^* - M_0$

$$R^* - M_0 = \{x^* \in R^* : |x^*| > r \text{ for all } r \in R_+ - \{0\}\}$$

In this framework of nomenclature, the infinitesimals are then S-finite nonstandard numbers, the only standard infinitesimal being zero.

If  $x^* \in R^*$  is S-finite, and nonstandard, then there is a unique standard number  $st(x^*)$  termed the standard part of  $x^*$  and is such that  $|x^* - st(x^*)| < e$  for  $e \in M_1$ . Then for an S-finite nonstandard number  $x^*$ ,  $st(x^*)$  is that standard number, which is uniquely infinitesimally close to  $x^*$ . One very often refers to the operation of "taking standard parts" and in that context  $st(\cdot)$  is conceived of as a mapping from  $R^*$  onto  $R$ . Salient features of the standard part map are that it is order preserving, and that it is homomorphic with respect to arithmetic operations. In sum,  $st(\cdot)$  satisfies:

$$(a) \quad st^{-1}(R) = R^*(\text{Mod } M_1)$$

$$(b) \quad (\forall x^* \in R^*) (\forall y^* \in R^*) \left\{ \begin{array}{l} (i) \quad st(x^* + y^*) = st(x^*) + st(y^*) \\ (ii) \quad st(x^* \cdot y^*) = st(x^*) \cdot st(y^*) \end{array} \right\}$$

Note that condition (a) implies the following:

(c) If, for  $x^*, y^* \in R^*$ ,  $st(x^*) = st(y^*)$ , then

$$(x^* = y^*)_{\text{Mod } M_1}.$$

Using this last statement, let us define the monad of a non-standard number  $x^* \in R^*$ , symbolized as  $u(x^*)$ , to be the set of points having the same standard part as  $x^*$ . Then a definition of the monad of  $x^*$  can be given as:

$$u(x^*) = \{y^* \in R^* : (x^* = y^*)_{\text{Mod } M_1}\}$$

We now give the promised result.

►Theorem: Let  $F^*$  be an internal set of the form  $[0, \omega] \subseteq N^*$  for  $\omega \in N^* - N$ . Then  $F^*$  has internal cardinality  $\|F^*\| = \omega$ , but has external cardinality  $|F^*| \geq 2^{N_0}$ .

Proof: (E. Zakon, op. cit., Theorem 3.1.)

Consider the unit interval in  $M^*$ ,  $[0, 1]^*$ , and consider the sets in  $[0, 1]^*$  defined by intervals with endpoints 0,  $1/\omega$ ,  $2/\omega$ , ...,  $\frac{(K^*+1)}{\omega}$ , ..., for  $K^* < \omega$  and  $K^* \in N^*$ . Then the subintervals  $[K^*/\omega, \frac{(K^*+1)}{\omega}]$  partition  $[0, 1]^*$ , a fact that can be seen from the analogous result in  $M$  that  $[0, 1] = \bigcup_{K < n} [K/n, \frac{(K+1)}{n}]$  for  $n \in N$  and  $K \in N$ . Then since  $\omega \in N^* - N$  and is infinite, the length of any interval,



$\text{len}[K^*/\omega, \frac{(K^*+1)}{\omega}] = [\frac{(K^*+1)}{\omega} - K^*/\omega] = 1/\omega$ , being the reciprocal of  $\omega$ , must be infinitesimal. However, since  $[0,1] \subseteq [0,1]^*$ , and the monads of each  $x \in [0,1]$  form a covering of  $[0,1]$ , which, moreover, is disjoint, i.e.,  $\left\{ \bigcup_{x \in [0,1]} u(x) \right\} \supseteq [0,1]$  and for distinct  $x, y \in [0,1]$ ,  $u(x) \cap u(y) = \emptyset$ , any covering of  $[0,1]^*$  by a disjoint infinitesimal family, a fortiori covers  $[0,1]$ . It is easy to see that the facts that  $[K^*/\omega, \frac{(K^*+1)}{\omega}]$  is a partition of  $[0,1]$  of infinitesimal length and the definition of  $u(x)$  for  $x \in [0,1]$ , imply that each interval  $[K^*/\omega, \frac{K^*+1}{\omega}]$  is contained in exactly one monad,  $u(x)$  for  $x \in [0,1]$ . In other words, the partition  $[K^*/\omega, \frac{K^*+1}{\omega}]$  is a refinement of some partition  $[K^*/\omega_1, \frac{K^*+1}{\omega_1}]$  in the monadic partition  $[u(x)]$ . Then the number of intervals in the family  $[K^*/\omega, \frac{K^*+1}{\omega}]$  cannot be less than the number of distinct monads  $u(x)$  for  $x \in [0,1]$ , and this has cardinality  $2^{N_0}$ , clearly.

But now the intervals  $[K^*/\omega, \frac{K^*+1}{\omega}]$  are in one-to-one correspondence with the values  $K^* \in [0, \omega - 1]$ , and the cardinality of these values is equal to that of the intervals, namely,  $2^{N_0}$ . This establishes the second assertion.

The first assertion of the theorem follows trivially from the definition of  $F^*$ .

Q.E.D.

## Nonstandard Measure Theory

The intuition embodied in Zakon's theorem can be carried a step further. Since by "counting" infinite internal sets, we include a "counting" of, say, the unit interval, shouldn't it be possible, at least conceptually, to construct a measure of  $[0,1]$  that is contained in a measure of  $F^* = [0,\omega]$ ? This intuition was answered in the affirmative in the paper by A. R. Bernstein and F. Wattenberg, "Nonstandard Measure Theory," in Applications of Model Theory to Algebra, Analysis and Probability Theory, W.A.J. Luxemburg, editor, Holt, Rhinehart and Winston, 1969. As their constructions are complicated, we will not develop their results in detail, but will satisfy the requirements of the introduction if we state the principal results without proof. In fact, we will not employ their construction, but rather the more recent and elegant generalizations of their results obtained by P. Loeb.

The Bernstein and Wattenberg construction of a measure on  $R^*$  was motivated by the desire to obtain an expression for the intuitively plausible proposition that  $\Pr(Q([0,\frac{1}{2}])) = \frac{1}{2}\Pr(Q([0,1]))$ , where  $\Pr(Q[0,1])$  is the probability of hitting a rational number with a dart-throwing mechanism on the interval  $[0,1]$ . Since, by Zakon's theorem, the interval  $[0,1]$  can be injected into an internal infinite  $^*\text{Finite}$  set,  $F^* = [0,\omega]$ , the gist of the Bernstein and Wattenberg

construction is to construct a measure on  $F^*$  that assigns the measure of a point to be an infinitesimal. Then since, in a manner of speaking,  $[0,1] \subseteq F^*$ , one can, in effect, "count," by means of  $\ast$ Finite sets, the number of rational points contained in any given interval. Moreover, for any interval, conceived of as an internal Lebesgue measurable set,  $A$ , then the Bernstein and Wattenberg approach yields a measure,  $u_{F^*}$ , such that  $(u_{F^*}(A^*) = \lambda(A))_{\text{Mod } M_1}$ , where  $\lambda(A)$  is the standard Lebesgue measure of  $A \subseteq [0,1]$ . Moreover,  $u_{F^*}$  is defined on all internal subsets of  $[0,1]$  and, thus, by projection onto  $M^*$ ,  $u_{F^*}$  yields a finitely additive measure for arbitrary sets of  $[0,1]$  that coincides with Lebesgue measure when it is defined. Their construction then provides a solution to the "Easy Problem of Measure" in  $R$ , first solved by Banach. The "Difficult Problem," which is not solvable in  $R$ , is to assign a nonnegative measure to every bounded interval  $E \subseteq R$  such that

- (i)  $E = [0,1] \Rightarrow \lambda(E) = 1$
- (ii)  $A = B + \{e\} \Rightarrow \lambda(A) = \lambda(B)$
- (iii)  $E = \bigcup_{j \in N} E_j$  and  $E_j \cap E_i = \emptyset \Rightarrow \lambda(E) = \sum_{j \in N} \lambda(E_j)$

The "Easy Problem" replaces (iii) with the restriction that  $j$  take values on a finite subset of  $N$ .

The actual approach of Bernstein and Wattenberg is to define a measure on the unit circle, conceived of as the real numbers modulo 1, and then to extend this into  $M^*$ , an  $N_1$ -

saturated enlargement. The extension of the unit circle is denoted as  $S^*$ . A  $\ast$ -finite subset of  $S^*$  is termed a sample. Any sample  $F^*$  has an associated sample measure which assigns to every subset  $A \subseteq S^*$  a nonstandard real number  $u_{F^*}(A)$ , where

$$u_{F^*}(A) = \frac{\|F^* \cap A\|}{\|F^*\|}.$$

► Theorem: (Bernstein and Wattenberg)

- (a)  $u_{F^*}(S^*) = 1$ ;  $u_{F^*}(\emptyset) = 0$ ;  $u_{F^*}(A) \geq 0$   $A \subseteq S^*$
- (b)  $A \subseteq B \Rightarrow u_{F^*}(A) \leq u_{F^*}(B)$
- (c) If  $\{A_j\}_{j \in N^*} \subseteq S^*$  and  $A_i \cap A_j = \emptyset$  if  $i \neq j$ , then  
 $\exists L^* \in N^*$  where  $j > L^* \Rightarrow u_{F^*}(A_j) = 0$  and  

$$u_{F^*}\left(\bigcup_{j \in N^*} A_j\right) = \sum_{j=1}^{L^*} u_{F^*}(A_j)$$

Note that in (c) the index  $j$  is over  $N^*$ .

If  $n^* \in N$  and  $F^*$  is a sample, then  $F^*$  is said to be  $n^*$ -invariant if and only if  $F^* = F^* + \frac{1}{n^*}$ , which is to say, that if  $x^* \in F^*$ , then  $x^* + \frac{1}{n^*} \in F^*$  (recall that addition on  $F^*$  is modulo 1).

► Theorem: (Bernstein and Wattenberg)

If  $F^*$  is  $n^*$ -invariant, then

- (a)  $A \subseteq S^* \Rightarrow u_{F^*}(A) = u_{F^*}\left(A + \frac{1}{n^*}\right)$
- (b)  $u_{F^*}\left(\left(a, a + \frac{t}{n^*}\right]\right) = t/n^*$  for  $t \in N^*$   $0 \leq t \leq n^*$
- (c)  $|u_{F^*}((a, b]) - (b - a)| \leq 1/n^*$



► Theorem: (Bernstein and Wattenberg)

If  $F^*$  is a sample which is  $n^*$ -invariant for every standard  $n^* \in N^*$  (i.e., every  $n \in N$ ), then there is an infinite  $P^* \in N^* - N$  for which  $K^* \leq P^*$ ,  $K^* \in N^*$  implies  $F^*$  is  $K^*$ -invariant.

The integer obtained in the above theorem is termed the mesh of  $F^*$ . To see that  $P^*$  must be in fact infinite, observe the following argument. Let  $T$  be the set of all integers in  $N^*$  such that  $F^*$  is  $K^*$ -invariant for any standard  $K^* \leq n$ , where  $n$  is standard and in  $T$ . Then it is easily verified that  $T$  is internal and must contain  $N$ . Let  $P^*$  be the first member of  $N^*$  that bounds  $T$ . Obviously  $P^*$  cannot be  $S$ -finite since  $N \subseteq T$ . Then  $P^*$  is infinite.

► Theorem: (Bernstein and Wattenberg)

Allow  $F^*$  to be a sample which is  $n$ -invariant (for every standard integer  $n^* \in N^*$ , and which has mesh  $P^*$ . Then,

- (a) If  $p, q \in N^*$ ,  $p \leq q \leq P^*$ ; then  $u_{F^*}((a, a + \frac{p}{q}]) = \frac{p}{q}$
- (b)  $|u_{F^*}((a, b]) - (b - a)| \leq \frac{1}{p^*}$ , where  $\frac{1}{p^*} \in M_1$

A sample  $F^*$  is termed a premeasure and the associated sample measure is termed a measure if the following is satisfied:

- (a)  $S^* \subseteq F^*$
- (b) For every  $n^* \in N^*$ , where  $n^*$  is standard,  $F^*$  is  $n^*$ -invariant
- (c) If  $A$  is Lebesgue measurable, then  $st(u_{F^*}(A^*)) = \lambda(A)$

► Theorem: (Bernstein and Wattenberg) Both premeasures and measures exist.

More recently, P. Loeb of the University of Illinois has elegantly generalized the result of Bernstein and Wattenberg to nonstandard representations of Borel measures. These, the so-called "Loeb spaces," have played an integral role in the recently obtained results concerning measurable nonstandard exchange economies by the Yale School of Mathematical Economics. Our use of Loeb's construction in Part II of the series was suggested by Professor Don Brown of the Cowles Foundation at Yale, and was first employed in the dissertations of his students, R. M. Anderson and Salim Rashid to obtain a nonstandard characterization of weak convergence in sequences of competitive exchange economies. Notably, Rashid obtained results in that framework that demonstrate a formal equivalence between the measure theoretic cores of Aumann and Hildenbrand and the cores of nonstandard exchange economies as developed in the works of Brown and Robinson.

### Standard Finite Cooperative Games

A finite cooperative game in the classic Von Neumann-Morgenstern sense is a pair,  $\Gamma(N, v)$ , where  $N$  is a finite set indexed by  $\{1, \dots, n\}$ , termed the players, and  $v$  is a set-valued function defined on  $P(N)$ , called the characteristic function, or, alternatively, the valuation of the game. The valuation function is assumed to satisfy the following:

- (1)  $v : P(N) \rightarrow R_+$
- (2)  $v(\{i\}) = 0$
- (3)  $v(\emptyset) = 0$  and  $v(N) < \infty$

It is frequently assumed, in addition, that  $v$  be superadditive, a condition which simplifies many characterizations of solution concepts. By superadditivity is meant:

- (4)  $v(S \cup T) \geq v(S) + v(T)$  if  $T \cap S = \emptyset$  for  $S, T \in P(N)$

A coalition structure for the game,  $\Gamma$ , is simply a partition of  $N$ ,  $B = \{B_1, \dots, B_m\}$ , where  $\bigcup_{j=1}^m B_j = N$  and if  $i \neq j$ ,  $B_i \cap B_j = \emptyset$ . An individually rational payoff configuration is a pair,  $(x, B)$ , where  $B$  is a fixed coalition structure, and  $x$  is a real valued function  $x : N \rightarrow R$  such that  $x(i) \geq v(\{i\}) = 0$  and  $x(B_j) = \sum_{i \in B_j} x(i) = v(B_j)$  for all  $B_j \in B$ . For the sake of consistency, one must also assume that  $v(B_j) \geq \sum_{i \in B_j} v(\{i\})$ .

Let  $T_{iK} = \{S \in P(N) : i \in S, \dots, K \notin S\}$ . For an i.r.p.c.  $(x, B)$ , an objection of  $i$  against  $j$  in  $B_j$  (it is assumed that

objections can only be made between players in the same element of the coalition structure) is a pair  $(y, S)$  where  $y \in (R_+)^{|S|}$  and  $S \in T_{ij}$  such that:

$$y(k) \geq x(k) \quad \text{for } k \in S$$

$$y(i) > x(i)$$

and

$$\sum_{j \in S} y(j) \leq v(S)$$

A counter-objection to  $(y, S)$  is a pair  $(z, D)$ , where  $z \in (R_+)^{|D|}$  and  $D \in T_{ji}$  such that:

$$z(t) \geq x(t) \quad \text{for } t \in D$$

$$z(t) \geq y(t) \quad \text{for } t \in S \cap D$$

and

$$\sum_{j \in D} z(j) \leq v(D)$$

An objection is said to be justified if there is no counter-objection to it.

The Bargaining Set,  $M_1^i(\Gamma)$ , is the set of all i.r.p.c.'s in  $(x, B)$  such that no justified objection can be made.

Define next, for  $S \subseteq N$ , the excess of the coalition  $S$  with respect to the i.r.p.c.  $(x, B)$  as  $e(S, x) = \left[ v(S) - \sum_{j \in S} x(j) \right]$ . Then, for  $i, j \in B_K$ , define  $S_{ij}(x) = \sup_{S \in T_{ij}} (e(S, x))$ . For a given  $B_K$ , and two players  $i, j \in B_K$ ,  $i$  is said to outweigh  $j$  with respect to  $(x, B)$  if  $S_{ij}(x) > S_{ji}(x)$  and  $x(j) > 0$ . Observe that if  $x(j) = 0$ ,  $j$  can effectively play alone since  $v(\{j\}) = 0$  and it is superfluous to  $j$  to be threatened. An



i.r.p.c.  $(x, \mathcal{B})$  is said to be balanced if there is no pair of players  $i, j \in B_K \in \mathcal{B}$ , such that  $i$  outweighs  $j$ . It follows that this condition is satisfied when  $[S_{ij}(x) - S_{ji}(x)]x(j) \leq 0$  for all  $i, j \in N$ .

The Kernel,  $K(\Gamma)$ , is the set of all balanced i.r.p.c.'s. For an arbitrary coalition structure  $\mathcal{B}$ , and i.r.p.c.  $(x, \mathcal{B})$ , consider the  $2^{|N|}$  dimensional vector  $\theta(x)$  obtained by arranging all excesses,  $v(S) - \sum_{j \in S} x(j)$ , for  $S \subseteq N$ , in non-increasing order. Then for  $x \in (x, \mathcal{B})$ ,

$$\begin{aligned} \theta(x) \text{df: } &= \left( \theta_1(x), \dots, \theta_{2^{|N|}}(x) \right) \\ &= \left( (v(S_1) - \sum_{j \in S_1} x(j)), \dots, (v(S_{2^{|N|}}) - \sum_{j \in S_{2^{|N|}}} x(j)) \right) \end{aligned}$$

and

$$\theta_i(x) \geq \theta_j(x) \quad \text{if } i < j$$

The Nucleolus is the set of all payoff configurations, i.r.p.e.'s  $x \in (x, \mathcal{B})$ , such that  $\theta(x)$  is minimal in the lexicographic order on  $(\mathbb{R})^{2^{|N|}}$ . If  $x, y \in (x, \mathcal{B})$ , then  $\theta(x) >_L \theta(y)$ , read  $\theta(x)$ , is lexicographically greater than  $\theta(y)$  if  $\exists i_0$  such that

$$\begin{aligned} \theta_i(x) &= \theta_i(y) \quad \text{for } i < i_0 \\ \theta_{i_0}(x) &> \theta_{i_0}(y) \end{aligned}$$

Then the Nucleolus,  $N(\Gamma)$ , is the set:

$$\{x \in (x, \mathcal{B}) : (\forall y \in (x, \mathcal{B})) \theta(x) \leq_L \theta(y)\}$$

For a given i.r.p.e.  $(x, \mathcal{B})$ , a coalition is said to be blocking if  $S \in P(N)$  and  $e(S, x) > 0$ . Then for a blocking

coalition, more is achievable acting separately than the members can obtain belonging to their respective units of  $B$ . With reference to the solution concept to be defined next, one usually allows  $B = N$ , since it is a solution concept involving coalitions and not alignments of individuals.

The Core,  $C(\Gamma)$ , is the set of all i.r.p.c.'s in  $(x, B)$  for which there is no blocking coalition. Then,

$$C(\Gamma) = \{x \in (x, B) \mid \sup_{S \in P(N)} [v(S) - \sum_{j \in S} x(j)] \leq 0\}$$

The following theorems we state without proof as they are well-known results in the literature. They serve to indicate the close relationships that the above solution concepts bear to each other and provide a basis for the inquiry that follows in the nonstandard context.

► Theorem I: For arbitrary coalition structures,

$$M_1^i(\Gamma) \neq \emptyset. \quad (B. Peleg)$$

► Theorem II: For arbitrary coalition structures,

$$K(\Gamma) \neq \emptyset. \quad (M. Maschler \text{ and } B. Peleg)$$

► Theorem III:  $K(\Gamma) \subsetneq M_1^i(\Gamma)$ . (M. Davis and M. Maschler)

For  $B = N$ , one has the following:

► Theorem IV:  $C(\Gamma) \subseteq M_1^i(\Gamma)$ .

► Theorem V: If  $C(\Gamma) \neq \emptyset$ , then  $K(\Gamma) \cap C(\Gamma) \neq \emptyset$ . (Davis and Maschler)

► Theorem VI:  $N(\Gamma) \neq \emptyset$ . (D. Schmeidler)

► Theorem VII:  $N(\Gamma) \subseteq K(\Gamma) \subseteq M_1^i(\Gamma)$ . (Schmeidler, Davis and Maschler)

► Theorem VIII: If  $C(\Gamma) \neq \emptyset$ , then  $N(\Gamma) \subseteq C(\Gamma)$ . (D. Schmeidler)

The other two major solution concepts for Standard Finite Cooperative Games, namely the Von Neumann-Morgenstern solution and the Shapley value, will not be treated in the present work.

## \*FINITE COOPERATIVE GAMES

Introduction

An extension of the classical games of the Von Neumann-Morgenstern variety, involving a finite number of participants and transferable utility, to a nonstandard \*Finite context is provided. By choosing the \*Finite set to be of the form  $[0, \omega]$ , for an infinite integer  $\omega \in N^* - N$ , the construction allows the treatment of denumerably infinite games by essentially finite means by imbedding them externally within the appropriate \*Finite set. In general, such games will not have non-empty Kernels, where the use of Kernel in this context is that of the syntactic equivalent of the Kernel for standard finite games. It then follows that solution concepts that are in close relationship to the Kernel, such as the Bargaining Set, and the Nucleolus, are in general not non-empty in their equivalent syntactic forms.

A related concept, that of the Quasi-Kernel,  $QK^*(\Gamma^*)$ , is shown to be, in general, non-empty in the nonstandard context for the class of \*Finite games with Q-bounded payoff. We provide an example in Section III which serves to indicate that  $QK^*(\Gamma^*)$  is a strictly weaker solution concept than the syntactic equivalent of the Kernel,  $K^*(\Gamma^*)$ . Additionally, we provide a discussion in Section IV, of related solution concepts to  $QK^*(\Gamma^*)$ .

Our existence proof is based in spirit on the construction originated by B. Peleg [3]. An additional feature of



the nonstandard \*Finite construction is found in its application to large economies in Mathematical Economics. Notably, the works of R. M. Anderson, Ph.D. dissertation, Yale, 1977, and Salim Rashid, Ph.D. dissertation, Yale, 1976, make use of such constructions (cf. Anderson and Rashid, "A Non-standard Characterization of Weak Convergence," Proceedings of the American Mathematical Society, Vol. 69, May 1978). In that context, the solution concepts treated in what follows can be employed to construct an interesting class of market games. Typically, some variety of "blocking" notion is employed to characterize stable outcomes. It can be shown that if the additional requirement is placed on the \*Finite game that it be S-non-atomic, then the blocking-type solutions treated in this work require that those coalitions that block be nonnegligible. In that spirit, John Geanakoplos has shown an equivalence between the nonstandard core and a coalitional form of the S-Bargaining Set treated in Section III (cf. "The Bargaining Set and Nonstandard Analysis," Harvard University paper, 1978, Center on Decision and Conflict in Complex Organizations).

Notation

$R$	The standard real numbers
$R_+$	The nonnegative real numbers (standard)
$N$	The standard natural numbers
$R^*$	The nonstandard real numbers
$N^*$	The extended natural numbers in $R^*$
$F^*$	An internal *Finite set in $N^*$ : $F^* \subseteq N^*$ and $\ F^*\  = \omega$
$\overset{>}{\underset{\uparrow}{\neq}}$	Greater than by at least an infinitesimal amount
$\overset{>>}{\underset{\uparrow}{\neq}}$	Greater than by a noninfinitesimal amount
$\geq$	Equal to or greater than by at least an infinitesimal amount
$(=)_{\text{Mod } M_1}$	Of at most an infinitesimal difference
$\omega$	An infinite integer
$\langle F^*, A(F^*), V^* \rangle$	A *Finite cooperative game for $F^* = [0, \omega]$ ; $v^* : (P(F))^* \rightarrow R^*$ ; $A(F^*)$ the algebra of internal subsets of $F^*$ , i.e. sets in $(P(F))^*$ . We assume that $v^*(\emptyset) = 0$ and that $v^*$ is superadditive on $A(F^*)$ with Q-bound
$(x^*, F^*)$	The set of payoff configurations for the game $\langle F^*, A(F^*), v^* \rangle$ . By definition, $(x^*, F^*) = \left\{ x^* \in (R_+^*)^{F^*} : \left[ \sum_{j \in F^*} x^*(j) = v^*(F^*) \right]_{\text{Mod } M_1} \right\}$

- $QK^*(\Gamma^*)$  The set of payoffs  $x^* \in (x^*, F^*)$  for the game  $\Gamma^*$  such that  $\left\{ \tilde{S}_{ij}^*(x^*) = \tilde{S}_{ji}^*(x^*) \right\}_{\text{Mod } M_1}$  a.e. in  $F^*$  where a.e. in  $F^*$  has the interpretation that the set in  $F^*$  for which  $\left\{ \tilde{S}_{ij}^*(x^*) \neq \tilde{S}_{ji}^*(x^*) \right\}_{\text{Mod } M_1}$  is negligible in the following sense,  $\left( \frac{\|S\|}{\|F^*\|} = 0 \right)_{\text{Mod } M_1}$
- $T_{ij}^*$   $\{S \in A(F^*) = i \in S \dots j \notin S\}$
- $\tilde{e}(S, x^*)$  The excess of  $S \in A(F^*)$  with respect to the payoff configuration  $x^* \in (x^*, F^*)$ . By definition,  $\tilde{e}(S, x^*) = v^*(S) - \sum_{j \in S} x^*(j)$
- $\tilde{S}_{ij}^*(x^*) = \max_{S \in T_{ij}^*} \tilde{e}(S, x^*)$
- $M_0$  The set of numbers in  $R^*$  that are finite -  $M_0 = \{x^* \in R^* : |x^*| < r \text{ for some } r \in R_+ - \{0\}\}$
- $M_1$  The set of numbers in  $R^*$  that are infinitesimal -  $M_1 = \{x^* \in R^* : |x^*| < r \text{ for all } r \in R_+ - \{0\}\}$
- S-topology The topology on  $R^*$  generated by the sets  $S(x^*, R) = \{y^* : |x^* - y^*| < r \text{ for some } r \in R_+ - \{0\}\}$
- Q-topology The topology on  $R^*$  generated by the sets  $Q(x^*, R) = \{y^* : |x^* - y^*| < r \text{ for some } r \in R_+^* - \{0\}\}$ . One observes that the Q-topology is stronger than the S-topology, i.e.  $S \subset Q$ .

- Q-closed (open) Closed (open) in the Q-topology.
- Q-bounded Bounded by  $r \in R_+^*$
- Q-convex hull For a set  $B \subseteq (R^*)^n$ ,  $n \in N^*$ ,  $Q\text{-con}(B)$  is the set of combinations  $\sum_{j=1}^n \alpha_j x_j^*$ ,  $\sum_{j=1}^n \alpha_j = 1$ ,  $\alpha_j \in R_+^*$ ,  $\alpha_j \in [0,1]$ ,  $x_j^* \in B$  for each  $j = 1, \dots, n$ . Then  $B$  is Q-convex if  $Q\text{-con}(B) \subseteq B$ .
- $u(x^*)$  The monad of a point  $x^* \in R^*$ . By definition,  

$$u(x^*) = \left\{ y^* \in R^* : (x^* = y^*) \text{ Mod } M_1 \right\}$$
- $st(x^*)$  The standard part of  $x^*$ . By definition,  

$$st(x^*) = \left\{ r \in R : (x = r) \text{ Mod } M_1 \right\}$$
 and is uniquely defined for  $x^* \in M_0$ .  $st(x^*)$  is not defined for  $x^* \notin M_0$ .



# I. \*Finite Games and the Quasi-Kernel

Let  $M$  be a set theoretical structure, sufficiently rich to generate the real number system and allow  $M^*$  to be an  $N_1$ -saturated enlargement of  $M$  in the sense of Robinson [4]. Then  $M^*$  generates a non-Archimedean extension of  $R, R^*$ , which is termed a nonstandard model of analysis. All references to nonstandard objects will be assumed to be with respect to  $M^*$ .

By a \*Finite cooperative game we will mean a triplet,  $\langle F^*, A(F^*), v^* \rangle = \Gamma^*$  where  $F^* = [0, \omega] \subseteq N^*$  is an internal \*Finite set,  $A(F^*)$  is the algebra of internal subsets of  $F^*$ , and  $v^*$  is a nonnegative  $Q$ -bounded set function from  $A(F^*)$  to  $R_+^*$  such that

- (i)  $v^*(\emptyset) = 0$
- (ii)  $v^*(F^*) \leq K^*$  for some  $K^* \in N^*$
- (iii)  $v^*(S \cup T) \geq v^*(S) + v^*(T)$  if  $S \cap T = \emptyset$

A pre-imputation is an internal assignment,  $x^* : F^* \rightarrow R^*$  such that  $\left\{ \sum_{j \in F^*} x^*(j) = v^*(F^*) \right\} \text{Mod } M_1$  and such that  $(\forall j \in F^*) (x^*(j) \in M_0^*)$ .

An imputation, or what we shall term a payoff configuration, is a pre-imputation that satisfies, in addition, individual rationality, that is,  $(\forall j \in F^*) (x^*(j) \geq 0)$ . We implicitly assume that the value of any single person coalition is null in this characterization of payoff configurations, that is,  $(\forall j \in F^*) (v^*({j}) = 0)$ . The set of payoff configurations will be denoted as  $(x^*, F^*)$ , assuming,

for the sake of simplicity, that the coalition structure is simply the grand coalition,  $F^*$ .

Definition I.1: We will define the notion of one player being stronger than another player in the game  $\Gamma^*$  in terms of the following concepts. Let  $i, j \in F^*$ ,  $i \neq j$ . Then we define

- (i)  $\bar{e}(S, x^*) = v^*(S) - \sum_{j \in S} x^*(j)$  for  $S \in A(F^*)$  and  $x^* \in (x^*, F^*)$
- (ii)  $T_{ij}^* = \{S \in A(F^*) : i \in S \text{ and } j \notin S\}$
- (iii)  $\bar{S}_{ij}^*(x^*) = \max_{S \in T_{ij}^*} \bar{e}(S, x^*)$  for  $x^* \in (x^*, F^*)$

Then a player,  $i$ , is said to be stronger than a player  $j$ , with respect to a given payoff configuration,  $x^* \in (x^*, F^*)$ , symbolized as  $i \supset j$  if it is the case that  $\bar{S}_{ij}^*(x^*) > \bar{S}_{ji}^*(x^*)$  and  $x^*(j) > 0$ . Two players,  $i$  and  $j$ , are said to be equipollent, symbolized as  $i \oplus j$ , if  $\sim(i \supset j)$  and  $\sim(j \supset i)$ ; that is, neither player outweighs the other.

Definition I.2: Let the following relation be defined with respect to  $\Gamma^*$  for  $\bar{x}^* \in (x^*, F^*)$ : For two players  $i, j \in F^*$ , let  $i \tilde{R}(\bar{x}^*) j$  mean that  $i \supset j$  with respect to  $\bar{x}^* \in (x^*, F^*)$ .

► Lemma I.3: The relation  $\tilde{R}(x^*)$  satisfies the following properties:

- (1)  $\tilde{R}(x^*)$  is acyclic.
- (2)  $\tilde{R}(x^*)$  is open in  $(x^*, F^*)$ . That is, the set  $\{y^* : i \tilde{R}(x^*) j\}$  is Q-open in  $(x^*, F^*)$  for  $i, j \in F^*$ .
- (3) If  $x^* \in (x^*, F^*)$  and  $x^*(j) = 0$  for some  $j \in F^*$ , then  $(\forall i \in F^*) (i \tilde{R}(x^*) j)$ .

Proof: Property (3) is straightforward.

Property (2) follows easily from the observation that for fixed  $S \in A(F^*)$ ,  $\tilde{e}(S, x^*)$  and therefore  $\tilde{S}_{ij}(x^*)$  is Q-continuous in  $x^*$ . The set  $\{x^* \in (x^*, F^*) : [\tilde{S}_{ij}^*(x^*) - \tilde{S}_{ji}^*(x^*)] \gg 0 \text{ and } x^*(j) \gg 0\}$  is obviously Q-open and for  $y^* \in u(x^*)$  taking  $x^*$  as fixed, it follows that  $[\tilde{S}_{ij}^*(y^*) - \tilde{S}_{ji}^*(y^*)] \gg 0$  and  $y^*(j) \gg 0$ , which gives the result.

To show acyclicity, property (1), it is sufficient to show that  $\tilde{R}(x^*)$  is transitive. The following proof is due to Maschler (unpublished Lecture Notes).

If  $i \tilde{R}(x^*) j$  and  $j \tilde{R}(x^*) k$ , then  $i \tilde{R}(x^*) k$  is to be shown. One has, therefore, by assumption, that  $[\tilde{S}_{ij}^*(x^*) \gg \tilde{S}_{ji}^*(x^*)]$  and  $[\tilde{S}_{jk}^*(x^*) \gg \tilde{S}_{kj}^*(x^*)]$  while  $x^*(j) \gg 0$  and  $x^*(k) \gg 0$ . It remains to show that  $[\tilde{S}_{ik}^*(x^*) \gg \tilde{S}_{ki}^*(x^*)]$ .

$$\text{Let } C = \left\{ S \in A(F^*) : S = \{ (T_{ij}^* \cup T_{kj}^*) \cup (T_{jk}^* \cup T_{ik}^*) \cup (T_{ki}^* \cup T_{ji}^*) \} \right\}$$

Let  $B \in A(F^*)$  such that  $(v^*(B) - \sum_{j \in B} x^*(j)) = \tilde{e}(B, x^*)$  is maximal in  $C$  under the inequality  $\geq$ .

Claim (1):  $(j \in B) \Rightarrow (i \in B)$

If not, then  $\bar{e}(B, x^*) = \bar{S}_{ji}^*(x^*)$ . By hypothesis, however,  $\bar{S}_{ij}^*(x^*) \underset{\uparrow}{\gg} \bar{S}_{ji}^*(x^*)$  and for some  $D \in C$ ,  $\bar{e}(D, x^*) \underset{\uparrow}{\gg} \bar{e}(B, x^*)$  which is false by definition of  $B$ .

Claim (2):  $(k \in B) \Rightarrow (j \in B)$

Same reasoning as claim (1).

Then  $k \notin B$  else for some  $S \in C$ , where  $i, j, k \in S$  which cannot be by construction of  $C$ . Also,  $i \in B$  else  $j \notin B$  and then  $k \notin B$  which contradicts the construction of  $C$  as well.

Then  $\bar{S}_{ik}^*(x^*) = \bar{e}(B, x^*) \underset{\uparrow}{\gg} \bar{S}_{ki}^*(x^*)$  and thus one has  $\bar{S}_{ik}^*(x^*) \underset{\uparrow}{\gg} \bar{S}_{ki}^*(x^*)$  while  $x^*(k) \underset{\uparrow}{> 0}$  which is by definition  $i \bar{R}(x^*) k$ .

Q.E.D.

In addition to the above properties we will make the assumption that the relation  $\bar{R}(x^*)$  is quasi-connected on the set of players. By quasi-connected we mean:  $(\forall x^* \in (x^*, F^*))$   
 $(\forall i \in F^*) (\exists j \in F^*) (i \bar{R}(x^*) j . v. j \bar{R}(x^*) i)$ . This assumption has the interpretation that there are no completely irrelevant players to the game, a player being irrelevant if no one is stronger than himself and he is stronger than no one. We require the assumption to avoid degenerate instances of stability in what follows. This assumption could conceivably be weakened to say that the set of irrelevant players is negligible.



Definition I.4: Corresponding to the relation  $\tilde{R}(x^*)$ , one has the following set:

$$\tilde{M}_i = \{x^* \in (x^*, F^*) : (\forall j \in F^*) (i \tilde{R}(x^*) j)\}$$

The following lemmas are adapted from Peleg [3].

► Lemma I.5: For any  $i \in F^*$ , then the following is true.

$$\tilde{M}_i = \{x^* \in (x^*, F^*) : (\forall j \in F^*) (i \tilde{R}^*(x^*) j)\} \neq \emptyset \text{ and is } Q\text{-closed.}$$

Proof: If  $\tilde{M}_i = \emptyset$ , then  $(\forall x^* \in (x^*, F^*)) (\exists j \in F^*) (i R(x^*) j)$ .

However, the set  $\{x^* \in (x^*, F^*) : \{x^*(j) = 0\} \subseteq x^*\}$  provides a contradiction by property (3) of Lemma I.3.

Let  $x^* \in \tilde{M}_i$ , then an equivalent form of  $\tilde{M}_i$  is  $\tilde{M}_i = \bigcap_{j \in F^*} \{y^* : \neg(i \tilde{R}(x^*) j)\}$ . Since  $F^*$  is internal, the intersection over  $F^*$ , of  $Q$ -closed sets is  $Q$ -closed. Therefore  $\tilde{M}_i$  is closed.

Q.E.D.

► Lemma I.6: If  $x^* \in (x^*, F^*)$ , then  $\exists i \in F^*$  such that  $x^* \in \tilde{M}_i$  and  $x^*(i) > 0$ .

Proof: If the first assertion is false, then  $x^* \in \left[ (x^*, F^*) - \left( \bigcup_{i \in F^*} \tilde{M}_i \right) \right]$  must be true. This implies that  $(\forall i \in F^*) (\exists j \in F^*) (i R(x^*) j)$ . By Lemma I.3  $\tilde{R}(x^*)$  is acyclic, and since  $F^*$  is internally \*Finite, one has that  $\left[ (\exists i \in F^*) (\forall j \in F^*) (\neg(i R(x^*) j)) \right]$ . This is contradictory.

If the second assertion were false, and  $x^*(i) = 0$  while  $x^* \in \tilde{M}_i$ , then  $(\forall j \in F^*) (j \tilde{R}(x^*) i)$ . But if  $x^* \in \tilde{M}_i$ , then  $(\forall j \in F^*) (i \tilde{R}(x^*) j)$ . Then by conjunction, one has  $(\forall j \in F^*) \left[ (\forall i \tilde{R}(x^*) j) \dots (\forall j \tilde{R}(x^*) i) \right]$ . In this case, however,  $\tilde{R}(x)$  is not Quasi-Connected.

Q.E.D.

Definition I.7: For each player  $i \in F^*$ , let the following function be defined:  $\tilde{c}_i : (x^*, F^*) \times \tilde{M}_i \rightarrow R_+^*$  where  $\tilde{c}_i(x^*) = d(x^*, \tilde{M}_i) = \inf_{y^* \in \tilde{M}_i} |x^* - y^*|$ .

► Lemma I.8:  $\tilde{c}_i(x^*) \geq 0$  and is Q-continuous.

Proof: Self-evident.

Q.E.D.

Definition I.9: A point  $\tilde{x}^* \in (x^*, F^*)$  is said to be Quasi-Balanced if  $\tilde{c}_i(x^*) = 0$  a.e. in  $F^*$ . We mean by a.e. in  $F^*$  that the set  $S \subseteq F^*$  for which  $\tilde{c}_i(x^*) \neq 0$  for  $i \in S$  is such that  $\left( \frac{\|S\|}{\|F^*\|} = 0 \right)_{\text{Mod } M_1}$ .

Alternatively put, a point  $\tilde{x}^* \in (x^*, F^*)$  is seen to be Quasi-Balanced if  $\tilde{x}^* \in \bigcap_{j \in F^* - S} \tilde{M}_j$  where  $S$  is such that  $\left( \frac{\|F^* - S\|}{\|F^*\|} = 1 \right)_{\text{Mod } M_1}$ . Then, by definition I.2, for any pair of players in  $(F^* - S)$ , with respect to  $\tilde{x}^* \in \bigcap_{j \in F^* - S} \tilde{M}_j$ , it must be the case that neither  $i \tilde{R}(\tilde{x}^*) j$ , nor  $j \tilde{R}(\tilde{x}^*) i$ ; in

brief, any two players in  $(F^* - S)$  are equipollent with respect to  $\bar{x}^*$ , if  $\bar{x}^*$  is Quasi-Balanced. The set of Quasi-Balanced payoff configurations in  $(x^*, F^*)$  for a \*Finite cooperative game  $\Gamma^*$ , we term the Quasi-Kernel, and use the symbol  $QK^*(\Gamma^*)$  to indicate the set,  $\{\bar{x}^* \in (x^*, F^*) : i \oplus j \text{ a.e. in } F^*\}$ .

An extremely useful alternative characterization of  $QK^*(\Gamma^*)$ , which we shall employ in subsequent sections, can be obtained from the following theorem, the reference for which we thank Dr. Lloyd Shapley of RAND.

► Theorem I.10: A payoff configuration  $\bar{x}^* \in (x^*, F^*)$  belongs to  $QK^*(\Gamma^*)$  if and only if  $(\bar{S}_{ij}^*(\bar{x}^*) = \bar{S}_{ji}^*(\bar{x}^*))_{\text{Mod } M_1} \text{ a.e. in } F^*$ .

Proof: We follow Maschler, Peleg, and Shapley [6] Theorem 2.7.

It is a relatively simple matter to show that the superadditivity of  $v^*(\cdot)$  on  $A(F^*)$  implies that  $\Gamma^*$  is S-monotonic, which is to say, that  $v^*(\cdot)$  on  $A(F^*)$  is such that  $v^*(S) \leq v^*(T)$  whenever  $S \subset T$ , for  $S, T \in A(F^*)$ .

Now if  $\bar{x}^* \in QK(\Gamma^*)$ , then  $(\forall i, j \in B) (i \oplus j)$  where  $B = (F^* - S)$  for some  $S \in A(F^*)$  such that  $\left[ \frac{\|S\|}{\|F^*\|} = 0 \right]_{\text{Mod } M_1}$ . Let those coalitions in  $B$ , not equal to  $B$  or  $\phi$ , such that  $\bar{e}(S, \bar{x}^*)$  is maximal under  $\geq$ , be denoted as  $D(\bar{x}^*)$ . Then,

$$D(\bar{x}^*) = \left\{ S \in (2^B - \{B, \phi\}) : \bar{e}(S, \bar{x}^*) \geq \bar{e}(T, \bar{x}^*) \text{ for } T \in (2^B - \{B, \phi\}) \right\}.$$

Let  $E = \cap \{S : S \in D(\bar{x}^*)\}$ , then by definition of  $D(\bar{x}^*)$ ,  $B \not\subseteq E$ . If  $E \neq \emptyset$ , then allow  $i \in E$  and  $\bar{j} \in B - E$ . Then clearly,  $\bar{S}_{i\bar{j}}^*(\bar{x}^*) > \bar{S}_{\bar{j}i}^*(\bar{x}^*)$ , and thus by Definition I.2,  $\bar{x}(j) = 0$ , since  $\bar{x}^* \in QK^*(\Gamma^*)$ . Let  $U$  be arbitrary and in  $D(\bar{x}^*)$ . Then,

$$\begin{aligned} \bar{e}(U, \bar{x}^*) &= v^*(U) - \sum_{j \in U} \bar{x}^*(j) = v^*(U) - \sum_{j \in B} \bar{x}^*(j) \\ v^*(U) - \sum_{j \in B} \bar{x}^*(j) &\leq v^*(B) - \sum_{j \in B} \bar{x}^*(j) \end{aligned}$$

and by S-monotonicity,

$$\bar{e}(U, \bar{x}^*) \leq v^*(B) - \sum_{j \in B} \bar{x}^*(j) \leq v^*(F^*) - \sum_{j \in F^*} \bar{x}^*(j) = 0$$

Then if  $\bar{j} \in B - U$ , then  $\bar{e}(\{\bar{j}\}, \bar{x}^*) = 0$ , and therefore  $\{\bar{j}\} \in D(\bar{x}^*)$ . Since then both  $\{\bar{j}\}$  and  $U$  are in  $D(\bar{x}^*)$ , it must be that  $E = \emptyset$ . Then the assumption that  $E \neq \emptyset$  is false and it must be the case that  $E = \emptyset$ , in fact.

Suppose next that for some  $i, \bar{j} \in B$  ( $\bar{S}_{i\bar{j}}^*(\bar{x}^*) \neq \bar{S}_{\bar{j}i}^*(\bar{x}^*)$ ) Mod  $M_1$ . Without loss of generality let this imply that  $\bar{S}_{i\bar{j}}^*(\bar{x}^*) > \bar{S}_{\bar{j}i}^*(\bar{x}^*)$  and that therefore since  $\bar{x} \in QK^*(\Gamma^*)$   $\bar{x}^*(\bar{j}) = 0$ . Then for some coalition in  $D(\bar{x}^*)$ , say  $U^0$ ,  $i \notin U^0$ , by virtue of the fact that  $E = \emptyset$ . Then, by S-monotonicity, one obtains

$$\begin{aligned} \bar{e}(U^0 \cup \{\bar{j}\}, \bar{x}^*) &= v^*(U^0 \cup \{\bar{j}\}) - \sum_{j \in U^0 \cup \{\bar{j}\}} \bar{x}^*(j) = v^*(U^0) - \sum_{j \in U^0} \bar{x}^*(j) \\ v^*(U^0 \cup \{\bar{j}\}) - \sum_{j \in U^0 \cup \{\bar{j}\}} \bar{x}^*(j) &\geq v^*(U^0) - \sum_{j \in U^0} \bar{x}^*(j) \\ &= \bar{e}(U^0, \bar{x}^*) \end{aligned}$$

By definition of  $\bar{S}_{\bar{j}i}^*(\bar{x}^*)$  and because  $U^0 \in D(\bar{x}^*)$ , it cannot be the case that  $\bar{S}_{\bar{j}i}^*(\bar{x}^*) < \bar{S}_{i\bar{j}}^*(\bar{x}^*)$  as supposed. Then  $(\bar{S}_{i\bar{j}}^*(\bar{x}^*) = \bar{S}_{\bar{j}i}^*(\bar{x}^*))$  Mod  $M_1$  for all  $i, j \in B$ , which is to say, a.e. in  $F^*$ ,



since by completely symmetric reasoning, one can show that  $\sim(\tilde{S}_{ji}^*(\bar{x}^*) \gg \tilde{S}_{ij}^*(\bar{x}^*))$  for a given  $\bar{j}, i \in B$ .

The other direction of the theorem is immediate since by definition, if  $(\tilde{S}_{ij}^*(\bar{x}^*) = \tilde{S}_{ji}^*(\bar{x}^*))_{\text{Mod } M_1}$  a.e. in  $F^*$ , then,  $i \oplus j$  a.e. in  $F^*$ .

Q.E.D.

► Theorem I.11: For a Q-bounded \*Finite cooperative game  $\Gamma^* = \langle F^*, A(F^*), v^* \rangle$ , there exists an  $\bar{x}^* \in (x^*, F^*)$  such that  $\bar{x}^* \in QK^*(\Gamma^*)$ .

Proof: Let the following mapping be constructed, which is Q-continuous by Lemma I.8:  $\tilde{f} : (x^*, F^*) \rightarrow (x^*, F^*)$  as

$$\tilde{f}(x^*)(j) = \frac{x^*(j) + \tilde{c}_j(x^*)}{1 + \sum_{j \in F^*} \tilde{c}_j(x^*)}$$

We will employ a version of a fixed point argument, to which the following lemmata are preliminary.

► Lemma I.11.1: The set of payoff configurations  $(x^*, F^*)$  is Q-convex.

Proof: We can proceed by internal \*Finite induction. For the case where  $n=2$ , consider  $x^*, y^* \in (x^*, F^*)$  and  $\alpha \in (0,1)$ .

We wish to show that

$$\left[ \alpha \left( \sum_{j \in F^*} x^*(j) \right) + (1-\alpha) \left( \sum_{j \in F^*} y^*(j) \right) = v^*(F^*) \right]_{\text{Mod } M_1}$$

By definition,  $\sum_{j \in F^*} x^*(j) = v^*(F^*) + e_1$  and  $\sum_{j \in F^*} y^*(j) = v^*(F^*) + e_2$  for  $e_1, e_2 \in M_1$ . Then, one obtains

$$\alpha \left( \sum_{j \in F^*} x^*(j) \right) + (1-\alpha) \left( \sum_{j \in F^*} y^*(j) \right) = \alpha(v^*(F^*) + e_1) + (1-\alpha)(v^*(F^*) + e_2).$$

However,  $\alpha(v^*(F^*) + e_1) + (1-\alpha)(v^*(F^*) + e_2) = v^*(F^*) + \alpha e_1 + (1-\alpha)e_2$ . But for  $e_1, e_2 \in M_1$ ,  $(\alpha e_1 + (1-\alpha)e_2) \in M_1$  for  $\alpha \in (0,1)$ . Then  $\alpha(v^*(F^*) + e_1) + (1-\alpha)(v^*(F^*) + e_2) = v^*(F^*) + e_3$  for  $e_3 \in M_1$ ,  $e_3 = \alpha e_1 + (1-\alpha)e_2$  and therefore one obtains

$\left( \alpha(v^*(F^*) + e_1) + (1-\alpha)(v^*(F^*) + e_2) = v^*(F^*) \right)_{\text{Mod } M_1}$ . But the last expression is simply that

$$\left( \alpha \left( \sum_{j \in F^*} x^*(j) \right) + (1-\alpha) \left( \sum_{j \in F^*} y^*(j) \right) = v^*(F^*) \right)_{\text{Mod } M_1}.$$

Assume the conclusion to be true for the case  $n-1$  for  $n \in N^*$ . Then choose  $\alpha_j \in (0,1)$  for  $j=1, \dots, n$  such that  $\sum_{j=1}^n \alpha_j = 1$ . Then by the hypothesis of the induction step,  $z^* = \left( \sum_{j=1}^{n-1} \alpha_j x_j^* \right) / (1 - \alpha_n)$  for  $x_j^* \in (x^*, F^*)$  is such that  $z^* \in (x^*, F^*)$ , since  $\sum_{j=1}^{n-1} \alpha_j / (1 - \alpha_n) = 1$ . But then, for  $x_n^* \in (x^*, F^*)$ , one observes that  $\sum_{j=1}^n \alpha_j x_j^* = \alpha_n x_n^* + (1 - \alpha_n) z^*$ . Then by the basis of the induction  $\left( \sum_{j=1}^n \alpha_j x_j^* \right) \in (x^*, F^*)$ .

Q.E.D.

► Lemma I.11.2: The set of payoff configurations is Q-closed.

Proof: It will suffice to show that  $(x^*, F^*)$  is closed under F-limits. If  $\phi: N^* \rightarrow (R^*)^{F^*}$  is a sequence on  $N^*$  with values in  $(R^*)^{F^*}$  then  $z^*$  is an F-limit of  $\phi$ , or  $\phi$  F-converges to  $z^*$

if it is the case that

$$(\forall \delta \in R_+ - \{0\}) (\exists n \in N) (\forall m \in N) (m \geq n \Rightarrow |\phi(m) - z^*| < \delta)$$

Then suppose  $\{x_j^*\}_{j \in N^*}$  is a sequence such that each  $x_j^* \in (x^*, F^*)$ , and let  $z^*$  be an  $F$ -limit of  $\{x_j^*\}_{j \in N^*}$ , but  $z^* \notin (x^*, F^*)$ . Then  $\left| \sum_{j \in F^*} z^*(j) \right| \notin u(v^*(F^*))$  which means that  $\left| \sum_{j \in F^*} z^*(j) - v^*(F^*) \right| > \delta$  for some  $\delta \in R_+ - \{0\}$ . But if  $z^*$  is an  $F$ -limit of  $\{x_j^*\}_{j \in N^*}$ , for  $m$  sufficiently large,  $x_m^* \in u(z^*)$ .

But in that case we obtain

$$\left| \sum_{j \in F^*} z^*(j) - v^*(F^*) \right| \leq \left| \sum_{j \in F^*} z^*(j) - \sum_{j \in F^*} x_m^*(j) \right| + \left| \sum_{j \in F^*} x_m^*(j) - v^*(F^*) \right|$$

and, therefore, that

$$\left| \sum_{j \in F^*} z^*(j) - v^*(F^*) \right| \leq e_1 + e_2 \quad \text{for } e_1, e_2 \in M_1^+.$$

This contradicts the assumption that  $z^* \notin (x^*, F^*)$ .

Q.E.D.

Then in the light of the above two lemmas, since  $(x^*, F^*)$  is  $Q$ -bounded, normalization by  $v^*(F^*)$  permits us to regard the set of payoff configurations as a  $Q$ -closed simplex of internal  $*Finite$  dimension in  $(R_+^*)^{F^*}$ . It is then possible to employ the following result.

► Lemma I.11.3: Let  $E$  be a  $*Finite$   $Q$ -closed simplex of internal dimension in  $(R_+^*)^{F^*}$ . Let  $\tilde{f}: E \rightarrow E$  be a  $Q$ -continuous mapping. For  $\hat{a} \in E$ , let  $A_{\hat{a}} = \{c(f(B(\hat{a}))) :$   
 $B = \{y \in E : |\hat{a} - y| < r \text{ for } r \in R_+^* - \{0\}\}\}$  where  $c(\cdot)$  denotes the  $Q$ -convex closure. Then for some  $\hat{a} \in E$ ,  $\hat{a} \in A_{\hat{a}}$ .

Proof: Machover and Hirschfeld, Lectures in Nonstandard Analysis, Springer Verlag, 1969.

Then Lemmas I.11.1, I.11.2, and I.11.3 with  $\tilde{f}$  defined as before, upon interpretation, allowing  $B$  to be defined as  $u(\hat{a})$ , where  $u(\hat{a})$  is the monad of  $\hat{a}$  defined as  $u(\hat{a}) = \bigcap_{\hat{a} \in U_v^*} U_v^*$ , and  $U_v^*$  is an open  $S$ -ball containing  $\hat{a}$ , one obtains

$\tilde{x}^* \in (x^*, F^*)$ , for which it is true that  $\tilde{x}^* \in A_{\tilde{x}^*} = \left\{ c \left( \tilde{f}((u(\tilde{x}^*))) \right) \right\}$ . Then  $\tilde{x}^* = \tilde{f}(y^*)$  for  $\tilde{f}(y^*) \in c \left( \tilde{f}(u(\tilde{x}^*)) \right)$ . The latter expression means that  $\left( \tilde{f}(y^*) = \tilde{f}(\tilde{x}^*) \right)_{\text{Mod } M_1}$ , from whence  $\left( \tilde{x}^* = \tilde{f}(\tilde{x}^*) \right)_{\text{Mod } M_1}$ . Then since

$$\tilde{f}(\tilde{x}^*)(j) = \frac{\tilde{x}^*(j) + \tilde{c}_j(\tilde{x}^*)}{1 + \sum_{j \in F^*} \tilde{c}_j(\tilde{x}^*)}$$

and  $\tilde{x}^*(j) \in M_0^+$  for any  $j \in F^*$ , it follows that

$$\left( \tilde{x}^*(j) = \frac{\tilde{x}^*(j) + \tilde{c}_j(\tilde{x}^*)}{1 + \sum_{j \in F^*} \tilde{c}_j(\tilde{x}^*)} \right)_{\text{Mod } M_1}$$

which equivalently stated is,

$$\left( \tilde{x}^*(j) \left( 1 + \sum_{j \in F^*} \tilde{c}_j(\tilde{x}^*) \right) = \tilde{x}^*(j) + \tilde{c}_j(\tilde{x}^*) \right)_{\text{Mod } M_1}$$

This last expression in turn implies

$$\left( \tilde{x}^*(j) \left( \sum_{j \in F^*} \tilde{c}_j(\tilde{x}^*) \right) = \tilde{c}_j(\tilde{x}^*) \right)_{\text{Mod } M_1}$$

which is true for  $\tilde{c}_j(\tilde{x}^*) \in M_1$ , for all  $j \in F^*$  only if

$\tilde{c}_j(\tilde{x}^*) > 0$  for  $j \in S \subseteq F^*$  such that  $\left( \frac{\|S\|}{\|F^*\|} = 0 \right)_{\text{Mod } M_1}$ , that is, only if  $\tilde{x}^*$  is Quasi-Balanced.

Q.E.D.



II. A \*Finite Game for which  $K^*(\Gamma^*) = \emptyset$  and  $QK^*(\Gamma^*) \neq \emptyset$

Allow the game  $\Gamma_N^*(F^*, v^*)$  to be defined as follows for

$F^* = [0, \omega]$ ,  $\omega \in N^* - N$

$$v^*(S) = \begin{cases} 1 & \text{for } S \in A(F^*) \text{ and } N(K) \subseteq S \\ 0 & \text{else} \end{cases}$$

By  $N(K)$  is meant the set  $\{n \in N : n \geq K\}$  for  $N$  the standard natural numbers and  $K \in N$ . Denote by  $K^*(\Gamma^*)$ , the syntactic equivalent of the Kernel for standard finite games in  $M^*$ . The following theorem serves to indicate that  $QK^*(\Gamma^*)$  is a strictly weaker concept than  $K^*(\Gamma^*)$  for \*Finite cooperative games.

► Theorem II.1: For the game  $\Gamma_N^*(F^*, v^*)$ ,  $K^*(\Gamma^*) = \emptyset$ .

Proof: Assume that  $\bar{x}^* \in K^*(\Gamma^*)$ . Then it must be the case that  $\sum_{j \in F^*} \bar{x}^*(j) = 1$ , since  $F^* \supseteq N$ . Therefore  $\bar{x}^* = (\bar{x}^*(1), \dots, \bar{x}^*(\omega))$  cannot be identically zero. While it may be true that  $\bar{x}^*(j) \underset{\uparrow}{>} 0$  for no  $j \in F^*$ , one can establish the following:

► Lemma II.1.2: There is a least standard integer  $\tilde{K} \in N$ , for which  $\bar{x}^*(\tilde{K}) \underset{\uparrow}{>} 0$ .

Proof: If not, then for some  $i \in F^* - N$ , and all  $j \in N$ ,  $\bar{x}^*(i) > \bar{x}^*(j)$ , else  $\bar{x}^* = 0$ . We show this leads to a contradiction.

Definition II.1.2.1: For a game  $\Gamma^*(F^*, v^*)$ ,  $K D \ell$ , read  $K$  is more desirable than  $\ell$ , if

$$v^*(S \cup \{K\}) \geq v^*(S \cup \{\ell\}) \text{ whenever } \ell, K \notin S \in A(F^*)$$

A payoff  $\bar{x}^* \in (x^*, F^*)$  is said to preserve desirability if  $K D \ell \Rightarrow \bar{x}^*(K) \geq \bar{x}^*(\ell)$ .

► Lemma II.1.2.2: If  $\bar{x}^* \in K^*(\Gamma^*)$ , then  $\bar{x}^*$  preserves desirability (Maschler and Peleg, 1966 [2]).

Proof: If  $\bar{x}^* \in K^*(\Gamma^*)$  and  $K D \ell$ , but  $\bar{x}^*(K) < \bar{x}^*(\ell)$ , choose  $S \in A(F^*)$  so that  $\tilde{S}_{\ell, K}^*(\bar{x}^*) = \bar{e}(S, \bar{x}^*)$ . Allow  $T = (S \cup \{K\} - \{\ell\})$ . Then  $v^*(T) \geq v^*(S)$  since  $K D \ell$ . Then because  $\bar{x}^*(K) < \bar{x}^*(\ell)$ , one has  $\bar{e}(T, \bar{x}^*) > \bar{e}(S, \bar{x}^*)$ . Then  $\tilde{S}_{K\ell}^*(\bar{x}^*) > \tilde{S}_{\ell K}^*(\bar{x}^*)$  and  $\bar{x}^*(\ell) > \bar{x}^*(K) \geq 0$ . But then  $K > \ell$  and therefore  $\bar{x}^* \notin K^*(\Gamma^*)$ .

Q.E.D.

To establish Lemma II.1.2, note that if for some  $i \in F^* - N$  and all  $j \in N$ ,  $\bar{x}^*(i) > \bar{x}^*(j)$ , by the contrapositive of Lemma II.1.2.2, for some  $S \in A(F^*)$  it must be that  $v^*(S \cup \{i\}) > v^*(S \cup \{j\})$ , where  $i, j \notin S$ . However, for  $\Gamma_N^*$ , any  $S$  not containing  $i, j$  has value 0 or 1, and that value is unchanged with or without either  $i$  or  $j$  for  $i \in F^* - N$  and  $j \in N$ . Therefore there can be no  $S \in A(F^*)$  such that  $v^*(S \cup \{i\}) > v^*(S \cup \{j\})$  for  $i, j \notin S$  and  $i \in F^* - N$  and  $j \in N$ .

Q.E.D.

► Lemma II.1.3:  $v^*(S)$  is superadditive.

Proof: If  $v^*(S)$  is not superadditive, then for some internal  $S \subseteq F^*$ ,  $S = S_1 \cup S_2$ ,  $S_1 \cap S_2 = \emptyset$ , it must be the case that  $v^*(S) < v^*(S_1) + v^*(S_2)$ .

It is either the case that  $N(K) \subseteq S$  for some  $K \in N$ , or  $N(K) \not\subseteq S$  for any  $K \in N$ .

If  $N(K) \not\subseteq S$  for any  $K \in N$ , then  $v^*(S) = 0$ . For  $v^*(S) < v^*(S_1) + v^*(S_2)$  to obtain, it must be the case that  $[v^*(S_1) \wedge v^*(S_2)] = 1$ . Then either  $S_1 \supseteq N(K_1)$  or  $S_2 \supseteq N(K_2)$  for  $K_1, K_2 \in N$ . Then  $S \supseteq N(K)$ ,  $K = K_1$  or  $K_2$ .

If  $S \supseteq N(K)$  for some  $K \in N$ , then  $v^*(S) = 1$ . For  $v^*(S) < v^*(S_1) + v^*(S_2)$  to obtain, then it must be the case that  $[v^*(S_1) \wedge v^*(S_2)] = 1$ . Then  $S_1 \supseteq N(K_1)$  and  $S_2 \supseteq N(K_2)$  for  $K_1, K_2 \in N$ . But then,  $N(K) \subseteq (S_1 \cap S_2)$  for  $\max\{K_1, K_2\} = K$ .

Q.E.D.

If  $\bar{x}^* \in K^*(\Gamma^*)$ , one requires, by virtue of the superadditivity of  $v^*(S)$ ,  $\bar{S}_{\bar{K}, \bar{K}+1}^*(\bar{x}^*) = \bar{S}_{\bar{K}+1, \bar{K}}^*(\bar{x}^*)$  by reasoning analogous to p. 17 footnote (\*\*) and Definition 3.1 with Theorem 3.3 of [2].

Obviously,  $[\sup_{S \in T_{\bar{K}, \bar{K}+1}^*} v^*(S)] = 0$ , while  $[\sup_{S \in T_{\bar{K}+1, \bar{K}}^*} v^*(S)] = 1$

where  $T_{\bar{K}+1, \bar{K}}^*$  is the collection of sets in  $A(F^*)$  containing  $\bar{K}+1$  but not  $\bar{K}$ . Then the requirement that  $\bar{S}_{\bar{K}+1, \bar{K}}^*(\bar{x}^*) = \bar{S}_{\bar{K}, \bar{K}+1}^*(\bar{x}^*)$  entails  $-\bar{x}^*(\bar{K}) = 1 - \sum_{j=\bar{K}+1}^{\omega} \bar{x}^*(j)$ . Equivalently, we have  $-2\bar{x}^*(\bar{K}) = 0$ . But this cannot be by choice of  $\bar{K}$  in Lemma II.1.2. This establishes the theorem.

Q.E.D.

On the other hand, for  $\tilde{x}^* \in QK^*(\Gamma^*)$ , it is required as a necessary condition that given  $\tilde{K}$  as above,

$$\left( \tilde{S}_{\tilde{K}, \tilde{K}+1}^* (\tilde{x}^*) = \tilde{S}_{\tilde{K}+1, \tilde{K}}^* (\tilde{x}^*) \right)_{\text{Mod } M_1} \quad \text{a.e. in } F^*, \text{ in which case,}$$

$$\left( -\tilde{x}^*(\tilde{K}) = 1 - \sum_{j=\tilde{K}+1}^{\omega} \tilde{x}^*(j) \right)_{\text{Mod } M_1} \quad \text{and therefore that}$$

$(-2\tilde{x}^*(\tilde{K}) = 0)_{\text{Mod } M_1}$  which for  $\tilde{x}^*(\tilde{K}) \underset{\uparrow}{>} 0$  does not entail a contradiction for  $\tilde{x}^*(\tilde{K}) \underset{\uparrow}{>} 0$ , a nonzero positive infinitesimal.

It can be easily verified that any symmetric vector

$$\tilde{x}^* = (1/n, \dots, 1/n) \text{ for } n \in N^* - N \text{ such that } \left( \frac{\|n\|}{\|\omega\|} = 1 \right)_{\text{Mod } M_1}$$

is in  $QK^*(\Gamma^*)$  for the game  $\Gamma_N^*$  defined as above.



### III. Solution Concepts Related to the Quasi-Kernel

In this section, we will develop solution concepts for \*Finite games, that are similarly defined in terms of coalitional excesses, that bear close relationships to the Quasi-Kernel. We begin with a solution concept that is analogous in form to that of the epsilon-core for standard finite games.

Recall from the previous section the game  $\Gamma_N^*(F^*, v^*)$ , whose characteristic function is

$$v^*(S) = \begin{cases} 1 & \text{for } S \in A(F^*) \text{ } N(K) \subseteq S \\ 0 & \text{else} \end{cases}$$

where  $N(K) = \{n \in N : n \geq K\} \subseteq N$ .

Definition III.1.1: Let  $C^*(\Gamma^*)$  be the set of payoffs in  $(x^*, F^*)$  such that  $\max_{S \in A(F^*)} \bar{e}(S, \bar{x}^*) \leq 0$ , and  $\sum_{j \in F^*} \bar{x}^*(j) = v^*(F^*)$ .

Definition III.1.2: Let  $QC^*(\Gamma^*)$  be the set of payoffs in  $(x^*, F^*)$  such that  $\max_{S \in A(F^*)} \bar{e}(S, x^*) \leq e$  a.e. in  $F^*$  for all  $e \in R_+ - \{0\}$ . Alternatively, we could term  $QC^*(\Gamma^*)$  as the least positive S-core, to reflect the fact that  $QC^*(\Gamma^*) =$

$$\bigcap_{e \in R_+ - \{0\}} C^e(\Gamma^*), \text{ where } C^e \text{ is the set, } \\ \left\{ x^* \in (x^*, F^*) : \max_{S \in A(F^*)} \bar{e}(S, x^*) \leq e \text{ a.e. in } F^* \text{ for } e \in R_+ - \{0\} \right\}.$$

In general,  $QC^*(\Gamma^*)$  is not non-empty.

The following two theorems serve to illustrate the distinction between  $C^*(\Gamma^*)$  and  $QC^*(\Gamma^*)$ .

► Theorem III.1.3: For the game  $\Gamma_N^*$ ,  $C^*(\Gamma_N^*) = \emptyset$ .

Proof: For  $\Gamma_N^*$ ,  $v^*(F^*) = 1$  since  $N \subseteq F^*$ . Suppose then that some  $\tilde{x}^* \in C^*(\Gamma_N^*)$ , then  $\sum_{j \in F^*} \tilde{x}^*(j) = 1$  and thus  $\tilde{x}^*$  is not identically zero.

We argue next, that for some least  $\tilde{K} \in N$ , it is true that  $\tilde{x}^*(\tilde{K}) > 0$ . Surely for some  $\tilde{K} \in F^*$ ,  $\tilde{x}^*(\tilde{K}) > 0$ , otherwise  $\tilde{x}^*$  would be identically zero. Since  $F^*$  is internal, there must be a first  $\tilde{K}$  such that  $\tilde{x}^*(\tilde{K}) > 0$  by Robinson [8] Theorem 3.1.7. Suppose  $\tilde{K} \in F^* - N$ . Then it is possible to choose  $\tilde{S} = [0, \tilde{K} - 1]$ . By choice of  $\tilde{K}$  and since  $N \subseteq \tilde{S}$ , clearly  $\tilde{e}(\tilde{S}, \tilde{x}^*) = \left(1 - \sum_{j \in \tilde{S}} \tilde{x}^*(j)\right) > 0$ . However, this contradicts that  $\tilde{x}^* \in C^*(\Gamma^*)$ . Therefore, for a least  $\tilde{K} \in N$ ,  $\tilde{x}^*(\tilde{K}) > 0$ .

Since  $\tilde{K} \in N$ , let  $S^* = [\tilde{K} + 1, \omega]$ . Since  $N(\tilde{K} + 1) \subseteq S^*$ ,  $v^*(S^*) = 1$  and therefore  $\tilde{e}(S^*, \tilde{x}^*) = \left(1 - \sum_{j \in S^*} \tilde{x}^*(j)\right) > 0$ . Then surely  $\max_{s \in A(F^*)} \tilde{e}(s, \tilde{x}^*) \geq \tilde{e}(S^*, \tilde{x}^*)$  and it is seen that it must be the case that  $C^*(\Gamma_N^*) = \emptyset$ .

Q.E.D.

The following game is closely related to  $\Gamma_N^*$ , and can be construed as a restricted version of  $\Gamma_N^*$  in the sense that winning coalitions in  $\Gamma_{N_1}^*$  are winning in  $\Gamma_N^*$  but not conversely. It is easily shown that  $C^*(\Gamma_{N_1}^*) = \emptyset$ . However, one does have the following.

► Theorem III.1.4: If the set of payoffs  $(x^*, F^*)$  are such that  $\sup_{j \in F^*} x^*(j) \in M_1$ , then the restriction of  $\Gamma_N^*$  to  $\tilde{\Gamma}_{N_1}^*$  with

characteristic function  $v^*(S) = 1$  if  $N(K) \subseteq S \in A(F^*)$  for some  $K \in N$  and  $\left( \frac{\|S\|}{\|F^*\|} = 1 \right)_{\text{Mod } M_1}$  and 0 otherwise, is such that  $QC^*(\Gamma_{N_1}^*) \neq \emptyset$ .

Proof: Consider the payoffs  $\bar{x}^* = (1/\bar{\omega}, \dots, 1/\bar{\omega})$  where  $((\bar{\omega}/\omega) = 1)_{\text{Mod } M_1}$ . If we denote these payoffs as  $\{\bar{x}^*\}$ , and if  $(\{\bar{x}^*\} \cap QC^*(\Gamma_{N_1}^*)) = \emptyset$ , then for some  $\tilde{S} \in A(F^*)$ ,  $\bar{e}(\tilde{S}, \bar{x}^*) > e$  for  $\bar{x}^* \in \{\bar{x}^*\}$  and some  $e \in R_+ - \{0\}$ . Clearly, we need only consider  $\tilde{S} \in A(F^*) : \left( \frac{\|\tilde{S}\|}{\|F^*\|} = 1 \right)_{\text{Mod } M_1}$ . Then  $\sum_{j \in \tilde{S}} \bar{x}^*(j) < 1 - e$  implies that it must be that  $\sum_{j \in F^* - \tilde{S}} \bar{x}^*(j) > e$  since  $\left( \sum_{j \in F^*} \bar{x}^*(j) = 1 \right)_{\text{Mod } M_1}$ , if  $\bar{x}^* \in (x^*, F^*)$ . However, if  $\left( \frac{\|\tilde{S}\|}{\|F^*\|} = 1 \right)_{\text{Mod } M_1}$ , then because  $\frac{\|\tilde{S}\|}{\|F^*\|} + \frac{\|F^* - \tilde{S}\|}{\|F^*\|} = 1$ ,  $\left( \frac{\|F^* - \tilde{S}\|}{\|F^*\|} = 0 \right)_{\text{Mod } M_1}$ . Since  $\sup_{j \in F^*} \bar{x}^*(j) \in M_1^+$ , then  $\left( \sum_{j \in F^* - \tilde{S}} \bar{x}^*(j) = 0 \right)_{\text{Mod } M_1}$ . Therefore  $\sim \left( \sum_{j \in F^* - \tilde{S}} \bar{x}^*(j) > e \right)$ , and therefore  $\sim \left( \sum_{j \in \tilde{S}} \bar{x}^*(j) < 1 - e \right)$ , and finally  $\sim(\bar{e}(\tilde{S}, \bar{x}^*) > e)$  for arbitrary  $e \in R_+ - \{0\}$ . Then  $\bar{x}^* \in QC^*(\Gamma_{N_1}^*)$ .

Q.E.D.

Definition III.1.5: Consider the set,  $C^e(\Gamma^*) = \{x^* \in (x^*, F^*) : \bar{e}(S, x^*) \leq e\}$  a.e. in  $F^*$ , for all  $S \in A(F^*)$ , for some  $e \in R_+ - \{0\}$ . For distinct pairs  $i, j \in F^*$ , let  $\delta_{ij}^e(\bar{x}^*) = \max\{\delta : \bar{x}^* - \delta e^i + \delta e^j \in C^e(\Gamma^*)\}$  for  $\bar{x}^* \in C^*(\Gamma^*)$  and  $e^i \in (R^*)^{F^*} \{e_j^i = 0, j \neq i \dots e_j^i = 1, j = i\}$ , while  $\delta \in R_+^*$ .

► Lemma III.1.6: If  $\bar{x}^* \in C^e(\Gamma^*)$ , then  $\delta_{ij}^e(\bar{x}^*) = e - \bar{S}_{ij}^*(\bar{x}^*)$  for  $i, j \in F^*$ ,  $i \neq j$  a.e. in  $F^*$ .

Proof: (Lemma 3.4 of [1])

If  $\bar{x}^*(i) > \delta((\bar{x}^*(i)/\delta) - 1)$  and  $\bar{x}^*(j) < \delta((\bar{x}^*(j)/\delta) + 1)$ , then

$$\begin{aligned} \bar{e}(S, \bar{x}^*) &< \bar{e}(S, \bar{x}^*(\delta)) & \text{if } S \in T_{ij}^* \\ \bar{e}(S, \bar{x}^*) &> \bar{e}(S, \bar{x}^*(\delta)) & \text{if } S \in T_{ji}^* \\ \bar{e}(S, \bar{x}^*) &= \bar{e}(S, \bar{x}^*(\delta)) & \text{if } S \in (T_{ij}^* \cup T_{ji}^*) \end{aligned}$$

where  $\bar{x}^*(\delta)$  is  $\bar{x}^*$  with  $\delta((\bar{x}^*(i)/\delta) - 1)$  and  $\delta((\bar{x}^*(j)/\delta) + 1)$  replacing  $\bar{x}^*(i)$  and  $\bar{x}^*(j)$ , respectively.

Allow  $\bar{S}$  to be in  $T_{ij}^*$  such that  $\bar{S}_{ij}^*(\bar{x}^*) = \bar{e}(S, \bar{x}^*)$ . Then if  $\bar{x}^* \in C^e(\Gamma^*)$ ,  $\bar{e}(\bar{S}, \bar{x}^*) + \delta_{ij}^e(\bar{x}^*) = e$ .

Q.E.D.

► Corollary III.1.7: If  $\bar{x}^* \in QC^*(\Gamma^*)$ , then

$$\begin{aligned} (\delta_{ij}^*(\bar{x}^*) = -S_{ij}(\bar{x}^*))_{\text{Mod } M_1} &\text{ a.e. in } F^*, \text{ where} \\ \delta_{ij}^*(\bar{x}^*) &= \max\{\delta \in R_+^* : \bar{x}^* - \delta e^i + \delta e^j \in QC^*(\Gamma^*)\}. \end{aligned}$$

Proof: Straightforward.

Q.E.D.

Definition III.1.8: Let  $\bar{x}^* \in C^e(\Gamma^*)$ . For each distinct pair  $i, j \in F^*$  denote by  $\bar{V}_{ij}(\bar{x}^*, e)$ , the interval  $[\bar{x}^* - D_{ij}, \bar{x}^* + D_{ji}]$ , for  $D_{ij} = \delta_{ij}^e(\bar{x}^*)e^i - \delta_{ij}^e(\bar{x}^*)e^j$  and  $D_{ji} = \delta_{ji}^e(\bar{x}^*)e^i - \delta_{ji}^e(\bar{x}^*)e^j$ .  $\bar{V}_{ij}(\bar{x}^*)$  will be called the variation of  $\bar{x}^*$  with respect to  $i$  and  $j$  in  $C^e(\Gamma^*)$ . (Compare Definition 3.6, p. 32 of [2].)



The variation of  $\bar{x}^*$  with respect to  $i$  and  $j$  is said to be symmetric in  $C^e(\Gamma^*)$  if  $D_{ij} = D_{ji}$ .

► Theorem III.1.9: If  $C^e(\Gamma^*) \neq \emptyset$ , then  $\bar{x}^* \in (C^e(\Gamma^*) \cap K^*(\Gamma^*))$  if and only if the variation of  $\bar{x}^*$  is symmetric in  $C^e(\Gamma^*)$  with respect to all distinct pairs  $i, j \in F^*$  a.e. in  $F^*$ .

Proof: (Theorem 3.7, p. 32 of [1]) Let  $\bar{x}^* \in C^e(\Gamma^*)$ . If  $\bar{x}^* \in K^*(\Gamma^*)$ , then  $\bar{S}_{ij}^*(\bar{x}^*) = \bar{S}_{ji}^*(\bar{x}^*)$  (superadditivity of  $v^*$  is implicit here). Then, by Definition IV.1.5, and Lemma IV.1.6,  $\delta_{ij}^e(\bar{x}^*) = \delta_{ji}^e(\bar{x}^*)$ . Since  $e$  is fixed,  $D_{ij} = D_{ji}$ .

The converse is obvious from Corollary III.1.7.

Q.E.D.

► Theorem III.1.10: If  $QC^*(\Gamma^*) \neq \emptyset$ , then  $(QC^*(\Gamma^*) \cap QK^*(\Gamma^*)) \neq \emptyset$ .

Proof: It will suffice to show that for some  $\bar{x}^* \in (x^*, F^*)$ ,  $\bar{V}_{ij}(\bar{x}^*)$  is symmetric in  $QC^*(\Gamma^*)$  a.e. in  $F^*$ , as this will entail  $(D_{ij} = D_{ji}) \text{ Mod } M_1$  a.e. in  $F^*$ .

Consider the restricted set of payoffs,  $(\widehat{x^*, F^*}) = (x^*, F^*) \cap QC^*(\Gamma^*)$ . By assumption,  $(\widehat{x^*, F^*}) \neq \emptyset$ . Furthermore, since  $QC^*(\Gamma^*)$  can easily be verified to be closed under  $F$ -limits, and to be  $Q$ -convex, since  $(x^*, F^*)$  is  $Q$ -bounded,  $(\widehat{x^*, F^*})$  is  $Q$ -bounded,  $Q$ -convex, and  $Q$ -closed. Then by Theorem I.11 there is a point  $\bar{x}^* \in (\widehat{x^*, F^*})$ , such that  $(\bar{S}_{ij}^*(\bar{x}^*) = \bar{S}_{ji}^*(\bar{x}^*)) \text{ Mod } M_1$  a.e. in  $F^*$ .

Corollary III.1.7 implies that if  $\tilde{x}^* \in QC^*(\Gamma^*)$ , then  $(\delta_{ij}^*(\tilde{x}^*) = -\tilde{S}_{ij}^*(\tilde{x}^*))_{\text{Mod } M_1}$  a.e. in  $F^*$ . However,  $\tilde{x}^* \in QK^*(\Gamma^*)$  means  $(\tilde{S}_{ij}^*(\tilde{x}^*) = \tilde{S}_{ji}^*(\tilde{x}^*))_{\text{Mod } M_1}$  a.e. in  $F^*$ . Therefore for some  $\tilde{x}^* \in (\widehat{x^*, F^*})$ ,  $(\delta_{ij}^*(\tilde{x}^*) = \delta_{ji}^*(\tilde{x}^*))_{\text{Mod } M_1}$  a.e. in  $F^*$ , implying  $(D_{ij} = D_{ji})_{\text{Mod } M_1}$  a.e. in  $F^*$ .

Q.E.D.

### III.2 The S-Bargaining Set, $SM_1^*(\Gamma^*)$

In this section, we show that a version of the classical Bargaining Set,  $M_1^i$ , developed by Aumann and Maschler for standard finite cooperative games can be defined for \*Finite cooperative games. We term this version of the Bargaining Set, the S-Bargaining Set to emphasize the specific requirement that any justified objection must yield a noninfinitesimal profit to the members aligned with the objector. We further require that any objector must be able to align himself with a nonnegligible coalition.

This latter feature of the S-Bargaining Set will follow perforce from the assumption that the game be S-non-atomic, since in that instance, the relevant payoffs to each individual will be no more than an infinitesimal by reasoning analogous to that found in Theorem 3.13 of Reference 2. The result is attributable in origin to Eugene Wesley.

Definition III.2.1: A \*Finite game is S-non-atomic if for any  $j \in F^*$ ,

$$\sup_{j \in S} [v^*(S) - v^*(S - \{j\})] \in M_1$$

Of course, since we are implicitly assuming that the characteristic function is superadditive, the latter expression is in fact

$$\sup_{j \in S} [v^*(S) - v^*(S - \{j\})] \in M_1^+$$

Included in the class of S-non-atomic \*Finite games are the Market games derived from Nonstandard Exchange Economies considered by Don Brown and Peter Loeb, "The Values of Non-standard Exchange Economies," Israel Journal of Mathematics, Vol. 25, 1976, and John Geanakoplos, "The Bargaining Set and Nonstandard Analysis," Harvard University, 1978, Discussion Paper. The S-Bargaining Set is, in fact, a stronger solution concept than the coalitional solution considered by Geanakoplos in the sense that the non-emptiness of the former implies the non-emptiness of the latter (cf. Geanakoplos, op. cit., Sections II, pp. 13-15, and III, pp. 6-9).

Definition III.2.2: Consider a \*Finite cooperative game to be as before,  $\Gamma^*(F^*) = \langle F^*, A(F^*), v^* \rangle$ , for  $F^* = [0, \omega]$  with payoffs  $(x^*, F^*) = \left\{ x^* \in (R_+^*)^{F^*} : \left[ \sum_{j \in F^*} x^*(j) = v^*(F^*) \right] \text{Mod } M_1 \right\}$ . For a fixed  $\bar{x}^* \in (x^*, F^*)$ , we say that a player  $i \in F^*$  has an (S)-objection to a player  $\bar{j} \in F^*$ , with respect to  $\bar{x}^*$ , if there exists a  $D \in T_{i\bar{j}}^*$  such that  $\left( \frac{\|D\|}{\|F^*\|} \neq 0 \right) \text{Mod } M_1$  for which there exists a vector  $\bar{z} \in (R_+^*)^D$  such that  $\sum_{j \in D} \bar{z}(j) \leq v^*(D)$  and  $\left( \sum_{j \in D} \bar{z}(j) - \sum_{j \in D} \bar{x}^*(j) \right) \gg 0$ . If  $i \in F^*$  has an (S)-objection to  $\bar{j} \in F^*$ , we denote the objection as  $(\bar{z}, D)$ .

A player  $\bar{j} \in F^*$  is said to have a counter-(S)-objection to the (S)-objection  $(\bar{z}, D)$  if there exists a  $T \in T_{\bar{j}i}^*$  such that  $\left( \frac{\|T\|}{\|F^*\|} \neq 0 \right) \text{Mod } M_1$  for which there exists a vector  $\bar{y} \in (R_+^*)^T$  such that  $\sum_{j \in T} \bar{y}(j) \leq v^*(T)$  and  $\bar{y}(j) \geq \bar{x}^*(j)$  for  $j \in (T - D)$  and  $\sum_{j \in T \cap D} \bar{y}(j) \geq \sum_{j \in T \cap D} \bar{z}(j)$ . If  $\bar{j} \in F^*$  has a counter-



(S)-objection to the (S)-objection  $(\bar{z}, D)$ , we denote the counter-(S)-objection as  $(\bar{y}, T)$ .

An (S)-objection is said to be justified if there is no counter-(S)-objection. If  $i \in F^*$  has a justified (S)-objection to  $j \in F^*$ , then we write  $i >_S j$ . We define the S-Bargaining Set,  $SM_1^*(\Gamma^*)$ , to be the set of payoffs in  $(x^*, F^*)$  for which there is no justified objection a.e. in  $F^*$ . Alternatively,  $SM_1^*(\Gamma^*) = \{x^* \in (x^*, F^*) : \neg(i >_S j) \dots \neg(j >_S i) \text{ a.e. in } F^*\}$ .

► Theorem III.2.3: For an S-non-atomic \*Finite cooperative game,  $\Gamma^* = \langle F^*, A(F^*), v^* \rangle$ ,  $QK^*(\Gamma^*) \subseteq SM_1^*(\Gamma^*)$ .

Proof: Let  $\bar{x}^* \in QK^*(\Gamma^*)$ . Then  $(\bar{S}_{ij}^*(\bar{x}^*) = \bar{S}_{ji}^*(\bar{x}^*))_{\text{Mod } M_1}$  a.e. in  $F^*$ . We show that for almost all players in  $F^*$ , any S-objection can be countered.

Suppose for some  $i$ , and some  $\bar{j}$ ,  $i >_S \bar{j}$  with respect to  $\bar{x}^*$ , by way of  $(\bar{z}, D)$ . Then  $\left(\frac{\|D\|}{\|F^*\|} \neq 0\right)_{\text{Mod } M_1}$  and clearly  $\bar{e}(D, \bar{x}^*) \gg 0$ . Since  $D \in T_{i\bar{j}}^*$ , it follows that  $\bar{S}_{i\bar{j}}^*(\bar{x}^*) \geq \bar{e}(D, \bar{x}^*)$ . Then, if  $(\bar{S}_{\bar{j}i}^*(\bar{x}^*) = \bar{S}_{i\bar{j}}^*(\bar{x}^*))_{\text{Mod } M_1}$ , for some  $T \in T_{\bar{j}i}^*$ ,  $\bar{S}_{\bar{j}i}^*(\bar{x}^*) = \bar{e}(T, \bar{x}^*)$ , and from the fact that  $(\bar{e}(T, \bar{x}^*) = \bar{S}_{i\bar{j}}^*(\bar{x}^*))_{\text{Mod } M_1}$ ,  $\bar{e}(T, \bar{x}^*) \gg 0$ . If  $(T \cap D) = \emptyset$ , and  $\left(\frac{\|T\|}{\|F^*\|} \neq 0\right)_{\text{Mod } M_1}$ , then  $(\bar{x}^*|_{T, T})$  is a counter-(S)-objection to  $(\bar{z}, D)$ . If  $(T \cap D) \neq \emptyset$  and  $\left(\frac{\|T\|}{\|F^*\|} \neq 0\right)_{\text{Mod } M_1}$ , we can construct the following vector in  $(R^*)^{F^*}$ :

$$\bar{y}(j) = \begin{cases} \bar{x}^*(j) & \text{if } j \in T - D \\ \bar{x}^*(j) + \frac{\bar{c}}{\|T \cap D\|} & \text{if } j \in T \cap D \end{cases}$$

If we allow  $\{\bar{c} = e(D, \bar{x}^*)\}_{\text{Mod } M_1}$ , from the fact that  $\bar{e}(T, \bar{x}^*) \geq \bar{e}(D, \bar{x}^*) \gg 0$ , it follows that  $\bar{e}(T, \bar{y}) \geq 0$ , or alternatively put, that  $\sum_{j \in T} \bar{y}(j) \leq v^*(T)$ . Further, since  $(\bar{z}, D)$  is an (S)-objection it must be that  $\bar{e}(D, \bar{z}) \geq 0$ , from whence it follows that  $\{(\bar{e}(D, \bar{x}^*) - \bar{e}(D, \bar{z})) = \bar{c}\}_{\text{Mod } M_1}$ . Then, since  $i \in D$  and  $i \notin T$ ,  $\|T \cap D\| < \|D\|$ , therefore that, for  $j \in T \cap D$ ,  $\bar{x}^*(j) + \frac{\bar{c}}{\|T \cap D\|} \geq \bar{x}^*(j) + \frac{\bar{c}}{\|D\|}$ . It can then be readily seen that in this case,  $(\bar{y}, T)$  is a counter-(S)-objection to  $(\bar{z}, D)$ .

If it should occur that  $T \in T_{ji}^*$  for which  $\bar{e}(T, \bar{x}^*) = \bar{S}_{ji}^*(\bar{x}^*)$  and  $(\bar{S}_{ji}^*(\bar{x}^*) = \bar{S}_{ij}^*(\bar{x}^*))_{\text{Mod } M_1}$  is negligible, i.e.  $\left(\frac{\|T\|}{\|F^*\|} = 0\right)_{\text{Mod } M_1}$ , we can employ the following argument to arrive at a contradiction.

Since  $D$  is nonnegligible,  $D - \{i\}$  must surely be non-negligible since  $\frac{\|\{i\}\|}{\|F^*\|} = \frac{1}{\omega}$ . Then, it is permissible to construct  $E = (T \cup (D - \{i\}))$ , and obviously  $\left(\frac{\|E\|}{\|F^*\|} \neq 0\right)_{\text{Mod } M_1}$ . Then since  $E \in T_{ji}^*$ , and  $\bar{S}_{ji}^*(\bar{x}^*) = \bar{e}(T, \bar{x}^*) = \max_{S \in T_{ji}} \bar{e}(S, \bar{x}^*)$ , it cannot be that  $\bar{e}(E, \bar{x}^*) \gg \bar{e}(T, \bar{x}^*)$ . Due to the superadditivity of  $v^*$ , we know that  $\bar{e}(E, \bar{x}^*) \geq \bar{e}(T, \bar{x}^*) + \bar{e}(D - \{i\}, \bar{x}^*)$ , and therefore that  $\sim(\bar{e}(E, \bar{x}^*) \gg \bar{e}(T, \bar{x}^*))$ , only if  $(\bar{e}(D - \{i\}, \bar{x}^*) = 0)_{\text{Mod } M_1}$ . However, by definition, we have the following expression:

$$\bar{e}(D, \bar{x}^*) = [v^*(D) - v^*(D - \{i\})] + \bar{e}(D - \{i\}, \bar{x}^*) - \bar{x}^*(i),$$

which implies that

$$\bar{e}(D, \bar{x}^*) + \bar{x}^*(i) = [v^*(D) - v^*(D - \{i\})] + \bar{e}(D - \{i\}, \bar{x}^*).$$

Then since  $(\bar{z}, D)$  is an (S)-objection and  $\bar{x}^*(i) \geq 0$ , we have

$[v^*(D) - v^*(D - \{i\})] + \bar{e}(D - \{i\}, \bar{x}^*) \geq 0$ . But  $v^*$  is S-non-atomic and therefore  $[v^*(D) - v^*(D - \{i\})] \in M_1^+$ , which implies that in the last expression,  $\bar{e}(D - \{i\}, \bar{x}^*) \geq 0$ . But in that case, we have that  $\bar{e}(E, \bar{x}^*) \geq \bar{e}(T, \bar{x}^*) + \bar{e}(D - \{i\}, \bar{x}^*) \geq \bar{e}(T, \bar{x}^*)$  and therefore that  $\bar{e}(E, \bar{x}^*) \geq \bar{e}(T, \bar{x}^*)$ .

Then if  $\bar{x}^* \in QK^*(\Gamma^*)$  and  $(\bar{S}_{ji}^*(\bar{x}^*) = \bar{S}_{ij}^*(\bar{x}^*))_{\text{Mod } M_1}$  a.e. in  $F^*$ , at most a negligible number of players cannot form counter-(S)-objections to (S)-objections raised against them in the manner prescribed above. It then follows that for any  $\bar{x}^* \in QK^*(\Gamma^*)$ ,  $\bigcup_{i \in F^*} \bigcup_{j \in F^*} \{(i, j) : (i >_S j) \dots (j >_S i)\} = S$ , such that  $\left( \frac{\|S\|}{\|F^*\|} = 0 \right)_{\text{Mod } M_1}$ . Then, any  $\bar{x}^* \in QK^*(\Gamma^*)$  is such that  $\bar{x}^* \in SM_1^*$ .

Q.E.D.

► Theorem III.2.4: For an S-non-atomic \*Finite cooperative game,  $\Gamma^* = \langle F^*, A(F^*), v^* \rangle$ ,  $SM_1^*(\Gamma^*) \neq \emptyset$ .

Proof: Theorems I.11 and III.2.3.

Q.E.D.

► Theorem III.2.5: For an S-non-atomic \*Finite cooperative game,  $\Gamma^* = \langle F^*, A(F^*), v^* \rangle$ ,  $QC^*(\Gamma^*) \subseteq SM_1^*(\Gamma^*)$ .

Proof: By Definition III.1.2, if  $\bar{x}^* \in QC^*(\Gamma^*)$ , then

$\max_{S \in A(F^*)} \bar{e}(S, \bar{x}^*) \leq e$  for any  $e \geq 0$ , a.e. in  $F^*$ . Clearly, no

S-objection can be made.

Q.E.D.

### III.3 The S-Nucleolus, $SN^*(\Gamma^*)$

In this section, we consider a nonstandard version of the solution concept developed by Schmeidler [5] and Kohlberg [4] for standard finite cooperative games, the Nucleolus. We term the nonstandard version of the Nucleolus, the S-Nucleolus, to emphasize the requirement of noninfinitesimal complaints on the part of significant coalitions to induce the appropriate quasi-order on the set of payoff configurations,  $(x^*, F^*)$ . We first consider versions of the standard Nucleolus in the  $^*Finite$  context.

Definition III.3.1: (Schmeidler [5]) Consider two payoffs  $\bar{x}_1^*, \bar{x}_2^* \in (x^*, F^*)$ , and define the relation  $\bar{x}_1^* <^* \bar{x}_2^*$  if

$$\begin{aligned} \max\{\bar{e}(S, \bar{x}_1^*)\} &< \max\{\bar{e}(S, \bar{x}_2^*)\} \\ \{S \mid \bar{x}_2^*(S) > \bar{x}_1^*(S)\} &\subset \{S \mid \bar{x}_1^*(S) > \bar{x}_2^*(S)\} \end{aligned}$$

where  $\bar{x}^*(S)$  is shorthand for  $\sum_{j \in S} \bar{x}^*(j)$ .

In a similar fashion, one defines the relations,  $\bar{x}_1^* \leq^* \bar{x}_2^*$  and  $\bar{x}_1^* =^* \bar{x}_2^*$ , by replacing the strict inequality between the two maxima with  $\leq$  and  $=$ , respectively.

Schmeidler's version of the Nucleolus can then be characterized as:

$$N_1^*(\Gamma^*) = \left\{ \bar{x}^* \in (x^*, F^*) : (\forall x^* \in (x^*, F^*)) (\bar{x}^* \leq^* x^*) \right\}$$

Definition III.3.2 (Kohlberg [4] and Bird [6]) Let the following set be defined for a given  $\bar{x}^* \in (x^*, F^*)$ :

$$Z(\bar{x}^*) = \left\{ z \in (R^*)^{F^*} : \bar{x}^* \in (x^*, F^*) \Rightarrow (\bar{x}^* + z) \in (x^*, F^*) \right\}$$

The set  $Z(\bar{x}^*)$  represents the set of all balanced transfers to the players in  $F^*$ , with respect to a given payoff  $\bar{x}^*$ .

The use of the term 'balanced' reflects the requirement that  $\left[ z(F^*) = \sum_{j \in F^*} z(j) = 0 \right]_{\text{Mod } M_1}$  for  $z \in Z(\bar{x}^*)$ . Kohlberg's version of the Nucleolus can then be characterized as:

$$N_2^*(\Gamma^*) = \left\{ \bar{x}^* \in (x^*, F^*) \left( \forall z \in Z(\bar{x}^*) \right) \left( \forall S \in A(F^*) \right) \left[ \left( \bar{e}(S, \bar{x}^*) > e \right) \right. \right. \\ \left. \left. \text{for } e \in R - \{0\} \dots (z(S) = 0) \right] \Rightarrow (\forall T \in A(F^*)) \left[ \left( \bar{e}(T, \bar{x}^*) > \right. \right. \right. \\ \left. \left. \left. \bar{e}(S, \bar{x}^*) \right) \Rightarrow (z(T) = 0) \right] \right] \right\}$$

Definition III.3.3: (Bird [6], Definition 3) An alternative and, as we will presently show, equivalent form of the Nucleolus, as defined by Schmeidler and Kohlberg, has been developed by C. Bird. Bird's version of the Nucleolus can be characterized as:

$$N_3^*(\Gamma^*) = \left\{ \bar{x}^* \in (x^*, F^*) : \max_{z \in Z(\bar{x}^*)} \{ \max_{\{S | z(S) > 0\}} \bar{e}(S, \bar{x}^*) - \max_{\{S | z(S) < 0\}} \bar{e}(S, \bar{x}^*) \} \leq 0 \right\}$$

► Theorem III.3.4:  $N_1^*(\Gamma^*) = N_2^*(\Gamma^*) = N_3^*(\Gamma^*)$

Proof: (Bird [6], Theorem 1)

$$(1) \quad N_1^*(\Gamma^*) \subseteq N_3^*(\Gamma^*)$$

If  $\bar{x}_1^* \in N_1^*(\Gamma^*)$ , then by Definition III.3.1,

$$(\forall x^* \in (x^*, F^*)) \left( \max_{\{S | x^*(S) > \bar{x}_1^*(S)\}} \bar{e}(S, \bar{x}_1^*) \leq \max_{\{S | \bar{x}_1^*(S) > x^*(S)\}} \bar{e}(S, x^*) \right)$$

Now, if  $\bar{x}_1^* \notin N_3^*(\Gamma^*)$ , then it must be the case that

$$(\exists \bar{z} \in Z(\bar{x}_1^*)) \left( \left( \max_{\{S | \bar{z}(S) > 0\}} \bar{e}(S, \bar{x}_1^*) - \max_{\{S | \bar{z}(S) < 0\}} \bar{e}(S, \bar{x}_1^*) \right) = c > 0 \right)$$



One notices that for a positive constant,  $T > 0$ ,

$\{S | \bar{z}(S) > 0\} = \{S | \frac{\bar{z}(S)}{T} > 0\}$ . Then, for a choice of  $T$  such that  $\frac{\|\bar{z}\| \cdot \omega}{T} < c/2$  where  $\omega = \|F^*\|$  and  $\|\bar{z}\| = \sup_{j \in F^*} |z(j)|$ ,

it can be seen that

$$\left( \max_{\{S | \frac{\bar{z}(S)}{T} > 0\}} \bar{e}(S, \bar{x}_1^*) - \max_{\{S | \frac{\bar{z}(S)}{T} < 0\}} \bar{e}(S, \bar{x}_1^* + \frac{\bar{z}}{T} \right) > 0$$

If we now define the payoff configuration  $\bar{x}^* = \bar{x}_1^* + \frac{\bar{z}}{T}$ , then it follows from the above that

$$\max_{\{S | \bar{x}^*(S) > \bar{x}_1^*(S)\}} \bar{e}(S, \bar{x}_1^*) > \max_{\{S | \bar{x}^*(S) > \bar{x}_1^*(S)\}} \bar{e}(S, \bar{x}^*)$$

However, this contradicts the fact that  $\bar{x}_1^* \in N_1^*(\Gamma^*)$ . Therefore  $\bar{x}_1^* \in N_3^*(\Gamma^*)$ .

$$(2) \quad N_3^*(\Gamma^*) \subseteq N_2^*(\Gamma^*)$$

If  $x_2^* \notin N_2^*(\Gamma^*)$ , then by Definition III.3.2,

$$\left( \exists \bar{z} \in Z(\bar{x}_2^*) \right) \left( \exists e \in \mathbb{R}_+ - \{0\} \right) \left[ \left( \forall S \in A(F^*) \right) \left( \left( \bar{e}(S, \bar{x}_2^*) \geq 0 \right) \vee \left( \bar{z}(S) \geq 0 \right) \right) \right. \\ \left. \left( \left( \bar{z}(T) > 0 \right) \wedge \left( \bar{e}(T, \bar{x}_2^*) = b > e \right) \right) \right]$$

Then surely,  $\max_{\{S | \bar{z}(S) > 0\}} \bar{e}(S, \bar{x}_2^*) \geq \bar{e}(T, \bar{x}_2^*) = b$ . However, since

$\bar{z}(S) \geq 0$  for any  $S \in A(F^*)$  such that  $\bar{e}(S, \bar{x}_2^*) \geq e$ , it follows that  $\max_{\{S | \bar{z}(S) < 0\}} \bar{e}(S, \bar{x}_2^*) < e$ . From the above, it is clear that

$$\max_{z \in Z(\bar{x}_2^*)} \left\{ \max_{\{S | z(S) > 0\}} \bar{e}(S, \bar{x}_2^*) - \max_{\{S | z(S) < 0\}} \bar{e}(S, \bar{x}_2^*) \right\} \geq b - e > 0$$

Therefore,  $\bar{x}_2^* \notin N_3^*(\Gamma^*)$ , and assertion (2) is established.

$$(3) \quad N_2^*(\Gamma^*) \subseteq N_1^*(\Gamma^*)$$

If  $\bar{x}_1^* \notin N_1^*(\Gamma^*)$ , then it is true by Definition III.3.1 that

$$\max_{\{S | \bar{x}_0^*(S) > \bar{x}_1^*(S)\}} \bar{e}(S, \bar{x}_1^*) > \max_{\{S | \bar{x}_1^*(S) > \bar{x}_0^*(S)\}} \bar{e}(S, \bar{x}_0^*) \geq \max_{\{S | \bar{x}_1^*(S) > \bar{x}_0^*(S)\}} \bar{e}(S, \bar{x}_1^*) = e$$

Let us now define  $\bar{z}^0 \in Z(\bar{x}_1^*)$  to be  $\bar{z}^0 = (\bar{x}_0^* - \bar{x}_1^*)$ . Then

clearly,  $\bar{z}^0(S) \geq 0$  when  $\bar{e}(S, \bar{x}_1^*) \geq e$ , and since

$$\max_{\{S | \bar{x}_0^*(S) > \bar{x}_1^*(S)\}} \bar{e}(S, \bar{x}_1^*) > e \text{ and if } \bar{x}_0^*(S) > \bar{x}_1^*(S) \text{ when } \bar{z}^0(S) > 0, \text{ for}$$

some  $T \in A(F^*)$ ,  $\bar{e}(T, \bar{x}_1^*) > e$  with  $\bar{z}^0(T) > 0$ . In that case,

however,  $\bar{x}_1^* \notin N_2^*(\Gamma^*)$  and assertion (3) holds. This completes proof of the theorem.

Q.E.D.

In view of the above theorem, it is permissible to refer to three versions of the Nucleolus collectively as  $N^*(\Gamma^*)$ .

We turn now to the definition of the S-Nucleolus.

Definition III.3.5: Consider two payoffs  $\bar{x}_1^*, \bar{x}_2^* \in (x^*, F^*)$ , then we define the relation  $\bar{x}_1^* <_S^* \bar{x}_2^*$  to mean that

$$\max_{\{S | \bar{x}_2^*(S) > \bar{x}_1^*(S)\}} \bar{e}(S, \bar{x}_1^*) << \max_{\{S | \bar{x}_1^*(S) > \bar{x}_2^*(S)\}} \bar{e}(S, \bar{x}_2^*)$$

Then for a \*Finite game  $\Gamma^* = \langle F^*, A(F^*), v^* \rangle$ , let the S-Nucleolus be defined as

$$SN^*(\Gamma^*) = \left\{ \bar{x}^* (x^*, f^*) : \max_{z \in Z(\bar{x}^*)} \left\{ \max_{\{S | z(S) > 0\}} \bar{e}(S, \bar{x}^*) - \max_{\{S | z(S) < 0\}} \bar{e}(S, \bar{x}^*) \right\} < e \right\}$$

for all standard  $e \in R_+ - \{0\}$ .

Intuitively, the requirement that  $\bar{x}^* \in SN^*(\Gamma^*)$  is that any balanced transfer in  $Z(\bar{x}^*)$ , that bestows a

noninfinitesimal credit to a coalition  $S \in A(F^*)$ , can do so only if that coalition's complaint under  $\bar{x}^*$  is not more than an infinitesimal more than the largest complaint lodged by any coalition receiving more than an infinitesimal debit.

### III.4 The Existence of the S-Nucleolus

In this section, we provide an existence proof of the S-Nucleolus as defined in the previous section. We begin with an alternative definition of the S-Nucleolus which is more in keeping with Schmeidler's original formulation of the Nucleolus for standard finite games.

**Definition III.4.1:** Let  $\Gamma^* = \langle F^*, A(F^*), v^* \rangle$  be a \*Finite cooperative game as defined in Section I. The algebra of internal subsets,  $A(F^*)$ , has internal cardinality  $2^\omega$ , where  $\omega = ||F^*||$ . For a fixed  $\bar{x}^* \in (x^*, F^*)$  let  $\bar{\theta}(\bar{x}^*) = [\bar{\theta}_1(\bar{x}^*), \bar{\theta}_2(\bar{x}^*), \dots, \bar{\theta}_{2^\omega}(\bar{x}^*)]$ , such that  $\bar{\theta}_j(\bar{x}^*) = \bar{e}(S_j, \bar{x}^*)$  for  $S_j \in A(F^*)$  and such that the components of  $\bar{\theta}(\bar{x}^*)$  are arranged in nonincreasing order, which is to say, that  $\bar{\theta}_i(\bar{x}^*) \geq \bar{\theta}_j(\bar{x}^*)$  if  $1 \leq i \leq j \leq 2^\omega$ .

Define the ordering  $\bar{\theta}(x^*) <_S \bar{\theta}(y^*)$  to be true of  $x^*, y^* \in (x^*, F^*)$ , if there is a subscript,  $i_0$ , such that

$$(\forall i < i_0) (\bar{\theta}_i(x^*) = \bar{\theta}_i(y^*)) \text{ Mod } M_1$$

and

$$\bar{\theta}_{i_0}(x^*) < \bar{\theta}_{i_0}(y^*)$$

As a straightforward consequence of Definitions III.3.5 and III.4.1, we have the following fact.

► **Proposition III.4.2:** For  $x^*, y^* \in (x^*, F^*)$ ,  $x^* <_S y^*$  if and only if  $\bar{\theta}(x^*) <_S \bar{\theta}(y^*)$ .

Alternatively, we can then redefine the S-Nucleolus as follows:

$$SN^*(\Gamma^*) = \left\{ \bar{x}^* \in (x^*, F^*) : \forall y^* \in (x^*, F^*) \sim (\bar{\theta}(y^*) <_S \bar{\theta}(x^*)) \right\}$$

or, expressing  $\sim(\bar{\theta}(y^*) <_S \bar{\theta}(x^*))$  as  $(\bar{\theta}(x^*) \leq_S \bar{\theta}(y^*))$ ,

$$SN^*(\Gamma^*) = \left\{ x^* \in (x^*, F^*) : \bar{\theta}(x^*) \leq_S \bar{\theta}(y^*) \forall y^* \in (x^*, F^*) \right\}$$

We consider next what we call the Centroid of a \*Finite cooperative game,  $\hat{C}(\Gamma^*)$ , and subsequently show that it is intimately related to the S-Nucleolus. Our work is based on the results of Maschler, Peleg, and Shapley [2], and the notion of Centroid is analogous to their concept of the Lexicographic Center.

Definition III.4.3: We define recursively two sequences,  $\{\bar{x}^j\}_{j=0}^{\bar{K}}$  and  $\{\tilde{\Sigma}^j\}_{j=0}^{\bar{K}}$ , of sets of payoffs and sets of coalitions, respectively, which is to say, that  $\bar{x}^j \subseteq (x^*, F^*)$  and  $\tilde{\Sigma}^j \subseteq A(F^*)$  for  $j = 0, 1, \dots, \bar{K}$  and  $\bar{K} \in N^*$ .

$\bar{x}^0 = (x^*, F^*)$  and  $\tilde{\Sigma}^0 = A(F^*) - \{F^* \cup \phi\}$ . On the assumption that for  $0 \leq j \leq \bar{K}$ ,  $\bar{x}^{j-1} \neq \phi$  and  $\tilde{\Sigma}^{j-1} \neq \phi$ , allow the following to be defined:

$$(a) \quad R^j = \text{st} \left( \min_{\bar{x}^* \in \bar{x}^{j-1}} \max_{S \in \tilde{\Sigma}^{j-1}} \bar{e}(S, \bar{x}^*) \right)$$

$$(b) \quad \bar{x}^j = \left\{ \bar{x}^* \in \bar{x}^{j-1} : \text{st} \left( \max_{S \in \tilde{\Sigma}^{j-1}} \bar{e}(S, \bar{x}^*) \right) = R^j \right\}$$

$$(c) \quad \tilde{\Sigma}_j = \left\{ S \in \tilde{\Sigma}^{j-1} : \text{st}(\bar{e}(S, \bar{x}^*)) = R^j (\forall \bar{x}^* \in \bar{x}^j) \right\}$$

$$(d) \quad \tilde{\Sigma}^j = \tilde{\Sigma}^{j-1} - \tilde{\Sigma}_j$$



Let  $\tilde{K}$  be the first value of  $j$  for which either  $\tilde{X}^j = \emptyset$  or  $\tilde{J}^j = \emptyset$ . We shall call the set  $\tilde{X}^{\tilde{K}}$  the Centroid of the game,  $\hat{C}(\Gamma^*) = \tilde{X}^{\tilde{K}}$ .

► Lemma III.4.4: Let  $\Gamma^*$  be a \*Finite cooperative game such that  $\Gamma^*$  is S-bounded, i.e.  $v^*(F^*) \in M_0^+$ . Then for some  $\tilde{K} \in N^*$ ,

- (1)  $\tilde{X}^{\tilde{K}} \neq \emptyset$  for some  $\tilde{K} \in N^*$
- (2)  $R^j$  is well defined for  $1 \leq j \leq \tilde{K}$
- (3)  $\tilde{X}^j$  is Q-convex and Q-closed for  $1 \leq j \leq \tilde{K}$
- (4)  $\tilde{J}^j \neq \emptyset$  for  $1 \leq j \leq \tilde{K}$
- (5)  $R^j < R^{j-1}$

Proof: Since  $\tilde{X}^0 = (x^*, F^*)$  and  $(x^*, F^*)$  is Q-convex and Q-closed, one can proceed by way of induction to establish (3), which, in the light of the fact that  $A(F^*)$  is an internal algebra, will imply (2) by compactness.

On the assumption that  $\tilde{X}^{j-1}$  is Q-convex, if  $\tilde{X}^j$  is not, then by (b) of Definition III.4.3 there exists  $\{\tilde{x}_t^*\}_{t=1}^n$ , such that  $\sum_{t=1}^n \alpha_t \tilde{x}_t^* = y^*$ , for  $\sum_{t=1}^n \alpha_t = 1$ ,  $\alpha_t \in (0, 1)$ , and  $\tilde{x}_t^* \in \tilde{X}^j$ , for  $t = 1, \dots, n$ , such that  $\max_{S \in \tilde{J}^{j-1}} \tilde{e}(S, y^*) \notin u(R^j)$ , where  $u(R^j)$  is the monad of  $R^j$  and  $R^j$  is a standard positive real. Then, by implication, one has that

$$\text{st}\left(\max_{S \in \tilde{J}^{j-1}} \tilde{e}(S, y^*)\right) = \text{st}\left(\max_{S \in \tilde{J}^{j-1}} \tilde{e}\left(S, \sum_{t=1}^n \alpha_t \tilde{x}_t^*\right)\right) \notin R^j$$

We will need the following sub-lemma, the demonstration of which, while elementary, we nonetheless provide.

► Lemma III.4.4.1: Let  $\{\tilde{x}_t^*\}_{t=1}^n \subset \tilde{X}^{j-1}$ . Then for  $S \in A(F^*)$

$$\tilde{e}(S, \sum_{t=1}^n \alpha_t \tilde{x}_t^*) = \sum_{t=1}^n \alpha_t \tilde{e}(S, \tilde{x}_t^*) \text{ for } \alpha_t \in (0,1), \sum_{t=1}^n \alpha_t = 1.$$

Proof: By definition,  $\tilde{e}(S, x^*) = v^*(S) - \sum_{j \in S} x(j)$ . We proceed by induction. For the basis of the induction, consider  $n=2$ .

Then,  $v^*(S) - \left[ \alpha \sum_{j \in S} \tilde{x}_1^*(j) + (1-\alpha) \sum_{j \in S} \tilde{x}_2^*(j) \right]$  is identical to

$$\alpha \left[ v^*(S) - \sum_{j \in S} \tilde{x}_1^*(j) \right] + (1-\alpha) \left[ v^*(S) - \sum_{j \in S} \tilde{x}_2^*(j) \right]$$

and therefore,  $\tilde{e}(S, \alpha \tilde{x}_1^* + (1-\alpha) \tilde{x}_2^*) = \alpha \tilde{e}(S, \tilde{x}_1^*) + (1-\alpha) \tilde{e}(S, \tilde{x}_2^*)$ .

Let us now assume that the hypothesis is true for  $n-1$ .

Then, for  $\alpha_t = 1, \dots, n, \alpha_t \in (0,1)$ , we have that

$$\sum_{t=1}^{n-1} \alpha_t \left[ v^*(S) - \sum_{j \in S} \tilde{x}_t^*(j) \right] / (1-\alpha_n) = \left[ v^*(S) - \sum_{t=1}^{n-1} \frac{\alpha_t}{1-\alpha_n} \sum_{j \in S} \tilde{x}_t^*(j) \right]$$

However, because  $\tilde{X}^{j-1}$  is convex,  $\left[ \sum_{t=1}^{n-1} \frac{\alpha_t}{1-\alpha_n} \tilde{x}_t^* \right] = z^* \in \tilde{X}^{j-1}$ , and thus for  $\tilde{x}_n^* \in \tilde{X}^{j-1}$ , by the basis of the induction, since

$((1-\alpha_n)z^* + \alpha_n \tilde{x}_n^*) = y^* \in \tilde{X}^{j-1}$ , we must have that

$$\alpha_n \left[ v^*(S) - \sum_{j \in S} \tilde{x}_n^*(j) \right] + (1-\alpha_n) \left[ v^*(S) - \sum_{j \in S} z^*(j) \right] = v^*(S) - \sum_{j \in S} y^*(j)$$

Therefore, since  $y^* = (1-\alpha_n)z^* + \alpha_n \tilde{x}_n^* = \sum_{t=1}^n \alpha_t \tilde{x}_t^*$  and by the hypothesis of the induction, it is true that

$$(1-\alpha_n) \left[ v^*(S) - \sum_{j \in S} z^*(j) \right] = \sum_{t=1}^{n-1} \alpha_t \left[ v^*(S) - \sum_{j \in S} \tilde{x}_t^*(j) \right]$$

Then

$$\sum_{t=1}^n \alpha_t \left[ v^*(S) - \sum_{j \in S} \tilde{x}_t^*(j) \right] = \left[ v^*(S) - \sum_{j \in S} y^*(j) \right] = v^*(S) - \left( \sum_{t=1}^n \alpha_t \sum_{j \in S} \tilde{x}_t^*(j) \right)$$

Q.E.D.

Then by the sub-lemma,  $\tilde{e}(S, x^*)$  is linear and therefore Q-convex, and from Definition III.4.3(a) we can conclude from the assumed convexity of  $\tilde{X}^{j-1}$ , that the set of minimizing values on  $\tilde{X}^{j-1}$  for  $\tilde{e}(S, x^*)$  with S fixed is convex. This contradiction implies that  $\tilde{X}^j$  is Q-convex.

An argument similar to that given in Lemma I.11.2 suffices to show that  $\tilde{X}^j$  is Q-closed, and we will not repeat it here. Then assertion (3) is established.

To establish (4), note that  $\tilde{\Sigma}_j = \emptyset$  for  $j < \bar{K}$  if and only if  $\tilde{\Sigma}^j = \tilde{\Sigma}^{j-1}$ . Then by Definition IV.4.3(c), for any  $S \in \tilde{\Sigma}^{j-1}$  there is an  $\tilde{x}_S^* \in \tilde{X}^j$  such that  $\text{st}(\tilde{e}(S, \tilde{x}_S^*)) < R^j$ . Then convexity allows the following construction to belong to  $\tilde{X}^{j-1}$ :

$$\tilde{x}_T^* = \frac{1}{\|\tilde{\Sigma}^{j-1}\|} \sum_{S \in \tilde{\Sigma}^{j-1}} \tilde{x}_S^*$$

Then, by assumption, if  $S \in \tilde{\Sigma}^{j-1}$ , it follows that

$$\text{st}(\tilde{e}(S, \tilde{x}_T^*)) = \text{st} \left( v^*(S) - \frac{1}{\|\tilde{\Sigma}^{j-1}\|} \sum_{S \in \tilde{\Sigma}^{j-1}} \left( \sum_{i \in S} \tilde{x}_S^*(i) \right) \right)$$

$$\text{st} \left( v^*(S) - \frac{1}{\|\tilde{\Sigma}^{j-1}\|} \sum_{S \in \tilde{\Sigma}^{j-1}} \left( \sum_{i \in S} \tilde{x}_S^*(i) \right) \right) = \text{st} \left( \frac{1}{\|\tilde{\Sigma}^{j-1}\|} \sum_{S \in \tilde{\Sigma}^{j-1}} \tilde{e}(S, \tilde{x}_S^*) \right)$$

$$\text{st} \left( \frac{1}{\|\tilde{\Sigma}^{j-1}\|} \sum_{S \in \tilde{\Sigma}^{j-1}} \tilde{e}(S, \tilde{x}_S^*) \right) = \frac{1}{\|\tilde{\Sigma}^{j-1}\|} \left( \sum_{S \in \tilde{\Sigma}^{j-1}} \text{st}(\tilde{e}(S, \tilde{x}_S^*)) \right) < R^j$$

This last inequality is in contradiction with Definition III.4.3(a), which establishes assertion (4).

Assertion (1) now follows from (4) in the light of the facts that the number of coalitions in  $A(F^*)$  is internally \*Finite, and for  $j \neq K$ ,  $\tilde{\Sigma}_j \cap \tilde{\Sigma}_i = \emptyset$ .

To establish (5), consider that for each  $S \in \tilde{\Sigma}^j$  there exists some  $\tilde{x}_S^*$  in  $\tilde{X}^j$  for which  $\text{st}(\tilde{e}(S, \tilde{x}^*)) < R^j$ . Then by the convexity of  $\tilde{X}^j$ ,  $x_T^* = \frac{1}{\|\tilde{\Sigma}^{j-1}\|} \sum_{S \in \tilde{\Sigma}^j} \tilde{x}_S^*$  is in  $\tilde{X}^j$  and is such that  $\text{st}(\tilde{e}(S, \tilde{x}_T^*)) < R^j$  when  $S \in \tilde{\Sigma}^j$ . Therefore, by Definition III.4.3(a),  $R^{j+1} \leq \text{st} \left( \max_{S \in \tilde{\Sigma}^j} \tilde{e}(S, \tilde{x}^*) \right) < R^j$ .

This concludes the proof of Lemma III.4.4.

Q.E.D.

► Lemma III.4.5: Let  $\Gamma^* = \langle F^*, A(F^*), v^* \rangle$  be an  $S$ -bounded \*Finite game. Then for  $1 \leq j \leq \bar{K}$ , if for  $\bar{x}_1^*, \bar{x}_2^* \in (x^*, F^*)$ ,  $\bar{x}_1^* \in \bar{X}^j$  and  $\bar{x}_2^* \in \bar{X}^{j-1} - X^j$ . Then  $\bar{\theta}(\bar{x}_1^*) <_S \bar{\theta}(\bar{x}_2^*)$ .

Proof: Construct the following partition of  $(P(F))^* - \{\phi \cup F^*\}$ ,  $\{\bar{L}_1, \dots, \bar{L}_{j-1}, \bar{L}^{j-1}\}$ . Then by Definition III.4.3(c) and Lemma III.4.4(5), we obtain

$$st(\bar{e}(S, x_1^*)) = R^h \geq R^{j-1}$$

for  $S \in \bar{L}_h$ ,  $h = 1, \dots, j-1$ . By Definition III.4.3(b), if  $S \in \bar{L}^{j-1}$ , then

$$st(\bar{e}(S, \bar{x}_1^*)) \leq R^j < R^{j-1} \text{ and } st(\bar{e}(S, \bar{x}_2^*)) \leq R^{j-1}$$

However,  $\bar{x}_2^* \notin \bar{X}^j$ , and thus  $R^j < st(\bar{e}(\bar{S}, \bar{x}_2^*)) \leq R^{j-1}$  for some  $\bar{S}$  in  $\bar{L}^{j-1}$ . Then, in Definition III.4.1, let  $i_0$  index  $\bar{S}$  in the ordering  $\bar{\theta}(\cdot)$ . Then,  $\bar{\theta}_{i_0}(\bar{x}_1^*) < \bar{\theta}_{i_0}(\bar{x}_2^*)$  and thus,  $\bar{\theta}(\bar{x}_1^*) <_S \bar{\theta}(\bar{x}_2^*)$ .

Q.E.D.

► Lemma III.4.6: Let  $\Gamma^* = \langle F^*, A(F^*), v^* \rangle$  be an  $S$ -bounded \*Finite game. Then  $st(X^{\bar{K}})$  is uniquely determined.

Proof: Since the values  $st(\bar{e}(S, \bar{x}^*))$  are constant on  $X^j$  if  $S \in \bar{L}^j$ , they are constant on  $\bar{X}^{\bar{K}}$ . In particular,  $st(\bar{e}(\{j\}, \bar{x}^*))$  is fixed for  $\forall j \in F^*$ .

Q.E.D.



► Theorem III.4.7: Let  $\Gamma^* = \langle F^*, A(F^*), v^* \rangle$  be a  $^*$ Finite S-bounded cooperative game. Then  $SN^*(\Gamma^*) \neq \emptyset$ .

Proof: Lemmas III.4.4, III.4.6, and III.4.5 yield that  $SN^*(\Gamma^*) = u[\text{st}(\hat{C}(\Gamma^*))]$  by Definition III.4.1, where  $u[\text{st}(\hat{C}(\Gamma^*))]$  is the monad of the standard part of the centroid.

Q.E.D.

Theorem III.4.7 serves to characterize the S-Nucleolus of an S-bounded  $^*$ Finite cooperative game as a unique infinitesimal region in  $(R_+^*)^{F^*}$ . Such a region has zero measure (Mod  $M_1$ ), in the product measure on  $(R_+^*)^{F^*}$ ,  $u_\omega \left( \prod_{j=1}^\omega (R_j^*) \right)$ ,  $R_j^* = R_+^*$  for  $j = 1, \dots, \omega$ . The existence of the S-Nucleolus for Q-bounded  $^*$ Finite games does not seem accessible by the above means, since in that context, one cannot assume that  $\text{st}(\tilde{e}(S, \tilde{x}^*))$  exists for  $S \in A(F^*)$ ,  $\tilde{x}^* \in (x^*, F^*)$ .

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### III.5 The Relation of the S-Nucleolus to Other Solution Concepts

As with the solution concepts treated in earlier sections that were related to the Quasi-Kernel, the S-Nucleolus is strictly weaker than the syntactic equivalent of the Nucleolus,  $N^*(\Gamma^*)$ , for \*Finite cooperative games in general. We illustrate this distinction by way of example.

► Theorem III.5.1: For the game  $\Gamma_N^*$  as defined in Section II,  $N^*(\Gamma_N^*) = \phi$ .

Proof: Suppose there exists some  $\bar{x}^* \in (x^*, F^*)$ , such that  $\bar{x}^* \in N^*(\Gamma_N^*)$ . Then since  $\left[ \sum_{j \in F^*} \bar{x}^*(j) = v^*(F^*) \right]_{\text{Mod } M_1}$ , and thus  $\bar{x}^*$  is not identically zero, there must be a least  $\bar{K} \in N$  such that  $\bar{x}^*(\bar{K}) \not\geq 0$ . Surely, there must be a first  $\bar{K} \in F^*$  such that  $\bar{x}^*(\bar{K}) \not\geq 0$ . If  $\bar{K} \in F^* - N$ , then since it is true that  $N^*(\Gamma^*) \subseteq K^*(\Gamma^*)$ , an argument identical to that of Lemma II.1.2 will suffice for the needed contradiction.

Next, let us construct the balanced transfer  $\bar{z}^* \in Z(\bar{x}^*)$  to be as follows:

$$\bar{z}^*(j) = \begin{cases} 0 & \text{if } j < \bar{K} \\ -\bar{x}^*(\bar{K}) & \text{if } j = \bar{K} \\ \frac{\bar{x}^*(\bar{K})}{\|F^* - \bar{K}\|} & \text{if } j > \bar{K} \end{cases}$$

where  $\bar{K}$  is the set  $\{n \in \mathbb{N} : n < \bar{K}\}$ . Clearly,

$$\left( \sum_{j \in F^*} \bar{z}^*(j) = 0 \right)_{\text{Mod } M_1}. \quad \text{Then one sees that it is true that}$$

$$\{S \in A(F^*) \mid \bar{z}(S) > 0\} \supseteq \{S \in A(F^*) \mid N(\bar{K} + j) \subseteq S, j = 1, 2, \dots\}$$

and that

$$\{S \in A(F^*) \mid \bar{z}(S) < 0\} = \{S \in A(F^*) \mid \bar{K} \in S \text{ and } (\exists j \in \mathbb{N}) (j > \bar{K} \dots j \notin S)\}.$$

It then follows that

$$\max_{\{S \mid \bar{z}^*(S) > 0\}} \bar{e}(S, \bar{x}^*) \geq \max_{\{S \mid \bar{z}^*(S) < 0\}} \bar{e}(S, \bar{x}^*)$$

Since  $v^*(S) = 0$  for  $S \in \{S \in A(F^*) \mid \bar{z}^*(S) < 0\}$  and since  $\bar{x}^*(\bar{K}) > 0$ ,  
and  $\bar{K} \in \{S \in A(F^*) \mid \bar{z}^*(S) < 0\}$ ,

$$\max_{\{S \mid \bar{z}^*(S) < 0\}} \bar{e}(S, \bar{x}^*) < 0$$

Then surely for  $z^* \in Z(x^*)$ , since  $\max_{\{S \mid \bar{z}^*(S) > 0\}} \bar{e}(S, \bar{x}^*) \geq 0$ ,

$$\max_{z^* \in Z(\bar{x}^*)} \left\{ \max_{\{S \mid \bar{z}^*(S) > 0\}} \bar{e}(S, \bar{x}^*) - \max_{\{S \mid \bar{z}^*(S) < 0\}} \bar{e}(S, \bar{x}^*) \right\} \geq \left\{ \max_{\{S \mid \bar{z}^*(S) > 0\}} \bar{e}(S, \bar{x}^*) - \max_{\{S \mid \bar{z}^*(S) < 0\}} \bar{e}(S, \bar{x}^*) \right\}$$

However, by choice of  $\bar{x}^*(\bar{K})$ ,  $\left\{ \max_{\{S \mid \bar{z}^*(S) > 0\}} \bar{e}(S, \bar{x}^*) - \max_{\{S \mid \bar{z}^*(S) < 0\}} \bar{e}(S, \bar{x}^*) \right\} > 0$

and by Definition III.3.3,  $\bar{x}^* \notin N_3^*(\Gamma_N^*)$  and therefore

$$\bar{x}^* \notin N^*(\Gamma_N^*).$$

Q.E.D.

Consider next the following game which is closely related to the game  $\Gamma_N^*$ . Define the game  $\Gamma_{N_2}^*(F^*, v^*)$  as

$$v^*(S) = \frac{\|S\|}{\|F^*\|} - \frac{1}{\|F^*\|} \quad \text{for } S \in A(F^*)$$

Then one observes that  $(v^*(S) = 1)_{\text{Mod } M_1}$  if  $\left( \frac{\|S\|}{\|F^*\|} = 1 \right)_{\text{Mod } M_1}$

and  $(v^*(S) = 0)_{\text{Mod } M_1}$  if  $\left( \frac{\|S\|}{\|F^*\|} = 0 \right)_{\text{Mod } M_1}$ . One also

observes that sets in  $A(F^*)$  that receive nonnegligible value



in  $\Gamma_{N_2}^*$ , also receive nonnegligible value in  $\Gamma_N^*$ , but not conversely. In an analogous fashion to the argument of Theorem III.5.1, one can show that  $N^*(\Gamma_{N_2}^*) = \emptyset$ . However, the following theorem, which is based on J. Grotte's treatment of the Central Game in [7], shows that the S-Nucleolus is non-empty for a larger class of \*Finite games than the syntactic equivalent of the Nucleolus for standard games. It thus represents a weaker solution concept.

► Theorem III.5.2: The set of payoffs  $\{\bar{x}^* \in (x^*, F^*) :$   
 $\bar{x}^* = (1/\bar{\omega}, \dots, 1/\bar{\omega}) \text{ for } ((\bar{\omega}/\omega) = 1)_{\text{Mod } M_1}\}$  is in  $SN^*(\Gamma_{N_2}^*(F^*))$ .

Proof: We recall that  $\omega = \|F^*\|$ . Then since for any  $S \in A(F^*)$  in the game  $\Gamma_{N_2}^*$ ,  $\bar{e}(S, \bar{x}^*) = \left( \frac{\|S\|}{\bar{\omega}} - \frac{1}{\bar{\omega}} \right) - \frac{\|S\|}{\bar{\omega}} = \left( \frac{\|S\|}{\bar{\omega}} - \frac{\|S\|}{\bar{\omega}} \right) - \frac{1}{\bar{\omega}}$ , and since  $\left( \frac{\bar{\omega}}{\omega} = 1 \right)_{\text{Mod } M_1}$ ,  $\left( \left( \frac{\|S\|}{\omega} - \frac{\|S\|}{\bar{\omega}} \right) = 0 \right)_{\text{Mod } M_1}$ ,  $\bar{e}(S, \bar{x}^*) \in u(-1/\omega)$ . Then, since any  $S \in A(F^*)$  is such that  $\bar{e}(S, \bar{x}^*) \in u(-1/\omega)$ , one arrives at

$$\left( \max_{\{z \in Z(\bar{x}^*)\}} \left\{ \max_{\{S | z(S) > 0\}} \bar{e}(S, \bar{x}^*) - \max_{\{S | z(S) < 0\}} \bar{e}(S, \bar{x}^*) \right\} = 0 \right)_{\text{Mod } M_1}$$

for any  $\bar{x}^* = (1/\bar{\omega}, \dots, 1/\bar{\omega})$  for  $((\bar{\omega}/\omega) = 1)_{\text{Mod } M_1}$ . By Definition III.3.5, any such  $\bar{x}^* \in (x^*, F^*)$  is in  $SN^*(\Gamma_{N_2}^*(F^*))$ .

Q.E.D.

Analogous to the relationships that the Kernel, Nucleolus, and Epsilon Core bear to one another in standard finite games, the Quasi-Kernel, S-Nucleolus, and Quasi-Core are interrelated for \*Finite cooperative games, as the following results show.

► Theorem III.5.3: Let  $\Gamma^*$  be a \*Finite cooperative game, if  $QC^*(\Gamma^*) \neq \emptyset$ , then  $SN^*(\Gamma^*) \subseteq QC^*(\Gamma^*)$ .

Proof: Definition III.1.2 and Definition III.3.5 with the remark following Proposition III.4.2.

Q.E.D.

► Theorem III.5.4: Let  $\Gamma^*$  be a \*Finite cooperative game, if  $\bar{x}^* \in SN^*(\Gamma^*)$ , then  $\bar{x}^* \in QK^*(\Gamma^*)$ , that is,  $SN^*(\Gamma^*) \subseteq QK^*(\Gamma^*)$ .

Proof: Let  $\bar{x}^* \in SN^*(\Gamma^*)$ , then we will show that

$\bar{s}_{ij}^*(\bar{x}^*) \not> \bar{s}_{ji}^*(\bar{x}^*)$  is not possible for any pair of players  $i, j \in F^*$ .

Suppose then that  $\bar{s}_{ij}^*(\bar{x}^*) > \bar{s}_{ji}^*(\bar{x}^*)$  for  $i, j \in F^*$ . Then  $\bar{s}_{ij}^*(\bar{x}^*) - \bar{s}_{ji}^*(\bar{x}^*) = d > 0$ . Since  $\bar{s}_{ij}^*(\bar{x}^*) = \max_{S \in T_{ij}^*} e(S, \bar{x}^*)$ , for some  $D \in T_{ij}^*$  and some  $E \in T_{ji}^*$ ,  $\bar{e}(D, \bar{x}^*) - \bar{e}(E, \bar{x}^*) = d$ . Let us then construct the balanced payoff transfer,

.

$$\bar{z}^*(j) = \begin{cases} \bar{x}^*(j) & \text{if } j \in F^* - \{E \cup D\} \\ \bar{x}^*(j) + c^- & \text{if } j \in E \\ \bar{x}^*(j) + c^+ & \text{if } j \in D \end{cases}$$

where  $c^- = -(d/2) \|E\|^{-1}$  and  $c^+ = +(d/2) \|D\|^{-1}$ . Then by construction  $\bar{z}^* \in Z(\bar{x}^*)$  and further  $\bar{z}^*(D) \underset{\uparrow}{\gg} 0$  and  $\bar{z}^*(E) \underset{\uparrow}{\ll} 0$ .

But then it is true that

$$\bar{z}^* \in Z(\bar{x}^*) : \max_{\substack{\{S | \bar{z}^*(S) \underset{\uparrow}{\gg} 0\}}} e(S, \bar{x}^*) \geq \bar{e}(D, \bar{x}^*) \underset{\uparrow}{\gg} \max_{\substack{\{S | \bar{z}^*(S) \underset{\uparrow}{\ll} 0\}}} \bar{e}(S, \bar{x}^*) \geq \bar{e}(E, \bar{x}^*)$$

However, in that case, by Definition III.3.5,  $\bar{x} \notin SN^*(\Gamma^*)$ .

Q.E.D.

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13. ABSTRACT <p>Solution concepts analogous to those found in the theory of finite cooperative games, namely, the Kernel, the Bargaining Set, the Epsilon-Core, and the Nucleolus, are defined for a class of Nonstandard Games. Such games are termed *Finite to indicate the use of certain infinite sets defined in the context of an <math>N_1</math>-saturated enlargement of the real number system. The principal solution concept, that of the Quasi-Kernel, is shown to be, in general, non-empty.</p> <p>*Presently at The Rand Corporation, Information Sciences Division, Santa Monica, California.</p>			

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